

Analysis of Binomial Coefficient Sequences Generated by Zeilberger Algorithm and Its Database

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Abstract

This is the final project of Combinatorics course at Rutgers University, Fall 2020. We generate the database of the binomial coefficient sum of three forms based on the Zeilberger algorithm, and we use Maple software to investigate their properties in the style of Frits Beuker's paper, [*Some Congruences for the Apery Numbers*](#)

1 Introduction

This paper is a final project of Combinatorics course at Rutgers University in Fall 2020 instructed by Professor Doron Zeilberger. In this project, we generate a database of all binomial coefficients sum of the form,

- $\sum (\binom{n}{k} * \binom{a_1 * n + b_1 * k}{k}, k=0..n) * x^k$ for all non-trivial $a_1, b_1, x \leq K$ for some fixed K ;
- $\sum (\binom{n}{k} * \binom{a_1 * n + b_1 * k}{k} * \binom{a_2 * n + b_2 * k}{k} * x^k, k=0..n)$ for all non-trivial $a_1, b_1, a_2, b_2, x \leq K$ for some fixed K ;
- $\sum (\binom{n}{k} * \binom{a_1 * n + b_1 * k}{k} * \binom{a_2 * n + b_2 * k}{k}, k=0..n) * \binom{a_3 * n + b_3 * n}{k} * x^k, k=0..n)$ for all non-trivial $a_1, b_1, a_2, b_2, a_3, b_3, x \leq K$ for some fixed K ;

Some of data can be found in the website <https://oeis.org/>, and we use this website to compare the available sequences to our database. In this project, we also study the properties of growth constant and the critical exponents of each sequence

Our Maple package and data files

We use Maple software to study this project, and we have the source code of Maple package and other supported files.

<https://github.com/yifanfyZ/Combinatorics-Project>

- For $x=2$

Sum(binomial(n, k)*2^k): a1=0, b1=1, OEIS: A000244

3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441, 1594323, 4782969, 14348907, ...

Sum(binomial(n, k)^2*2^k): a1=1, b1=0, OEIS: A001850

3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, 44642381823, ...

*Same as $x=1$, $a_1=1$, $b_1=1$

Sum(binomial(n, k)*binomial(2*k, k)*2^k): a1=0, b1=2, OEIS: A084771

5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, ...

*Also named as the coefficients of expansion of $1/\sqrt{1-10*x+9*x^2}$

Sum(binomial(n, k)*binomial(n+k, k)*2^k): a1=1, b1=1, OEIS: A006442

5, 37, 305, 2641, 23525, 213445, 1961825, 18205345, 170195525, 1600472677, 15122515985, 143457011569, 1365435096485, 13033485491077, 124715953657025, ...

*Also named as the expansion of $1/\sqrt{1-10*x+x^2}$

Sum(binomial(n, k)*binomial(2*n, k)*2^k): a1=2, b1=0, OEIS: A026000

5, 41, 377, 3649, 36365, 369305, 3800305, 39490049, 413442773, 4354393801, 46082942185, 489658242241, 5220495115997, 55818956905529, 598318746037217, ...

*Also named as $a(n)=T(2n,n)$, where T is the Delannoy triangle.

2.2 Database of Binomial Coefficient Sum of The Second Form

Here, we use the function $\text{InfoA2}(A,n,k,N,L1,r)$ in the Maple package, where A is the input as a form of $A=[[a_1,b_1],[a_2,b_2],x]$, and a_1, b_1, a_2, b_2, x are positive integers, and a small positive integer r. The output will list N terms of the coefficient with the growth constant, the critical exponent and the estimated asymptotic to order of r. And we will explore the database of $(a_1, b_1, a_2, b_2) = \{(0, 0, 1, 1), (1, 1, 0, 0), (10, 1, 0, 0), (0, 1, 2, 3), (3, 0, 3, 0)\}$ for $x=1$ and $x=2$ with first 15 terms.

- For $x = 1$

Sum(binomial(n, k)): a1=0, b1=1, a2=0, b2=1, OEIS: [A000079](#)

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, ...

Sum(binomial(n, k)^3): a1=1, b1=0, a2 =1, b2=0, OEIS: [A000172](#)
2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260, 278415920, 2046924400,
15148345760, 112738423360, 843126957056, ...

Sum(binomial(n, k)^2*binomial(10n, k)): a1=10, b1=0, a2=1, b2=0, OEIS: *Not in the OEIS - This is A Candidate for Submission*
11, 271, 8246, 278191, 9960011, 370484578, 14152208480, 551288912047, 21800220231764,
872396185820271, 35250583283675320, 1435812888898655938, 58878785488489939815,
2428402057368816190016, 100656580396709054419496, ...

Sum(binomial(n, k)*binomial(2n+3k, k)): a1=0, b1=1, a2=2, b2=3, OEIS: *Not in the OEIS - This is A Candidate for Submission*
6, 60, 681, 8156, 100661, 1266531, 16151064, 208009460, 2699376252, 35241625845,
462347547030, 6090267736523, 80497299641206, 1067054270924002,
14180111476352636, ...

Sum(binomial(n, k)*binomial(3n, k)^2): a1=3, b1=0, a2=3, b2=0, OEIS: *Not in the OEIS - This is A Candidate for Submission*
10, 298, 1188, 465338, 20515760, 939210040, 44145376656, 2115776170426,
102935384406748, 5067855779140048, 251930600336230540, 12624549933207755720,
636913613472524512420, 32318150503381365325504, 1648060116216438577090688, ...

■ For x = 2

Sum(binomial(n, k)): a1=0, b1=1, a2=0, b2=1, OEIS: A000244
3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441, 1594323, 4782969,
14348907, ...

Sum(binomial(n, k)^3): a1=1, b1=0, a2 =1, b2=0, OEIS: *Not in the OEIS - This is A Candidate for Submission*
3, 21, 171, 1521, 14283, 138909, 1385163, 14072193, 145039923, 1512191781, 15914734443,
168802010001, 1802247516891, 19350710547021, 208783189719531, ...

Sum(binomial(n, k)^2*binomial(10n, k)): a1=10, b1=0, a2=1, b2=0, OEIS: *Not in the OEIS - This is A Candidate for Submission*

21, 921, 48681, 2840481, 176092821, 11362336185, 753932487609, 51060899002113, 3512520847057077, 244613354000672121, 17204702866984169481, 1220018788242348544737, 87109992818329664964789, 6256181062992267103354137, 451585908830088913597950681, ...

Sum(binomial(n, k)*binomial(2n+3k, k)): a1=0, b1=1, a2=2, b2=3, OEIS: *Not in the OEIS - This is A Candidate for Submission*

11, 209, 4487, 101553, 2367811, 56274161, 1355400015, 32968699745, 808017951803, 19922464707409, 493603863412759, 12279036302805393, 306494959668199027, 7672549762108085105, 192549186645770584927, ...

Sum(binomial(n, k)*binomial(3n, k)^2): a1=3, b1=0, a2=3, b2=0, OEIS: *Not in the OEIS - This is A Candidate for Submission*

19, 1045, 72487, 5574897, 454639539, 38505863341, 3348686813855, 296967529358721, 26734377377511667, 2435605453611819205, 224052060167404956055, 20776634976893766571665, 1939704908436380759774067, 182138184516359778031213821, 17188174017764716498068548847, ...

2.3 Database of Binomial Coefficient Sum of The Third Form

Here, we use the function #InfoA3(A,n,k,N,L1,r) in the Maple package, where A is the input as a form of A=[[a1,b1],[a2,b2],[a3,b3],x], and a1, b1, a2, b2, a3, b3, x are positive integers, and a small positive integer r. The output will list N terms of the coefficient with the growth constant, the critical exponent and the estimated asymptotic to order of r. And we will explore the database of (a1, b1, a2, b2, a3, b3) = {(1, 0, 1, 0, 1, 0), (2, 0, 2, 0, 2, 0), (10, 0, 1, 0, 2, 0), (0, 1, 1, 0, 2, 0), (3, 0, 3, 0, 3, 0)} for x=1 and x=2 with first 15 terms.

- For x = 1

Sum(binomial(n, k)^4): a1=1, b1=0, a2=1, b2=0, a3=1, b3 =0, OEIS: *Not in the OEIS - This is A Candidate for Submission*

2, 18, 164, 1810, 21252, 263844, 3395016, 44916498, 607041380, 8345319268, 116335834056, 1640651321764, 23365271704712, 335556407724360, 4854133484555664, ...

Sum(binomial(n, k)*binomial(2n, k)^3): a1=2, b1=0, a2=2, b2=0, a3=2, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

9, 345, 18774, 1179225, 80504259, 5806240986, 435305257200, 33587212215897,
2649602504341125, 212734715333273595, 17326968991675900830,
1428146995932615353850, 118899062798952252031614, 9984016451703686659087200,
844595170736130768472713024, ...

Sum(binomial(n, k)^2*binomial(10n, k)*binomial(2n, k)): a1=10, b1=0, a2=1, b2=0, a3=2, b3=0,
OEIS: *Not in the OEIS - This is A Candidate for Submission*

21, 1461, 141546, 16041141, 1983727521, 259325655330, 35241876991944,
4929106681207989, 704890373488893255, 102598698665927348961,
15149615152883586283974, 2263796450371099675837026,
341692957129022350245778041, 52019000920374240623926044456,
7978278402650427805286069950296, ...

Sum(binomial(n, k)^2*binomial(2n, k)) a1=0, b1=1, a2=1, b2=0, a3=2, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

3, 23, 210, 2103, 22253, 244094, 2746404, 31490615, 366388077, 4312707273, 51247387854,
613774801374, 7399935424170, 89724381787868, 1093260213845960, ...

Sum(binomial(n, k)*binomial(3n, k)^3) a1=3, b1=0, a2=3, b2=0, a3=3, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

28, 3808, 734860, 165611264, 40751148528, 10612167717304, 2875254953626440,
802160396510054400, 228888162251222317984, 66487448169190985667808,
19595511719590624902974764, 5845134215593480801785845144,
1761283893790602138235209401680, 535325768444259358601333048357208,
163926538867048444515501187708017360, ...

■ For x = 2

Sum(binomial(n, k)^4 a1=1, b1=0, a2=1, b2=0, a3=1, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

3, 37, 495, 7761, 131283, 2336629, 43174911, 819869185, 15906350403, 313905320037,
6281740700271, 127173173346129, 2599950664710675, 53601450936173877,
1113117091905581055

Sum(binomial(n, k)*binomial(2n, k)^3): a1=2, b1=0, a2=2, b2=0, a3=2, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

17, 1121, 105797, 11638657, 1394871257, 176703090929, 23274019853437,
 3155287329808385, 437393769682065185, 61714090469890625281,
 8833688532199400538037, 1279617713549144507310721, 187233900417071575702815977,
 27632393388235069407035057297, 4108427761623636232828580096237, ...

Sum(binomial(n, k)^2*binomial(10n, k)*binomial(2n, k)): a1=10, b1=0, a2=1, b2=0, a3=2, b3=0,
 OEIS: *Not in the OEIS - This is A Candidate for Submission*

41, 5201, 887741, 176331841, 38334555641, 8836734229409, 2121511497498637,
 524724774175062017, 132771829040490128489, 34205380390045607027601,
 8941673074909033010844509, 2365846530143754113755275073,
 632363224687925097594766386681, 170494896845496886027992046062881,
 46313377028022633155161165420215341, ...

Sum(binomial(n, k)^2*binomial(2n, k)) a1=0, b1=1, a2=1, b2=0, a3=2, b3=0, OEIS:
Not in the OEIS - This is A Candidate for Submission

5, 57, 809, 12577, 206565, 3517785, 61457625, 1094273793, 19771649189, 361430293657,
 6670081948425, 124068704570209, 2323173855659909, 43749208935909177,
 827929777224958329, ...

Sum(binomial(n, k)*binomial(3n, k)^3) a1=3, b1=0, a2=3, b2=0, a3=3, b3=0, OEIS:
 Not in the OEIS - This is A Candidate for Submission

55, 14365, 5305879, 2288247729, 1077640811615, 537172498208125, 278611641900025895,
 148806870474428382721, 81290869424486131861639, 45209271728818418009595325,
 25510751401115284755776524375, 14569579518186331498278023388465,
 8405688001358431113570702536150703, 4891676692416450121074492165578006013,
 2868058621571120919070978186306790993639, ...

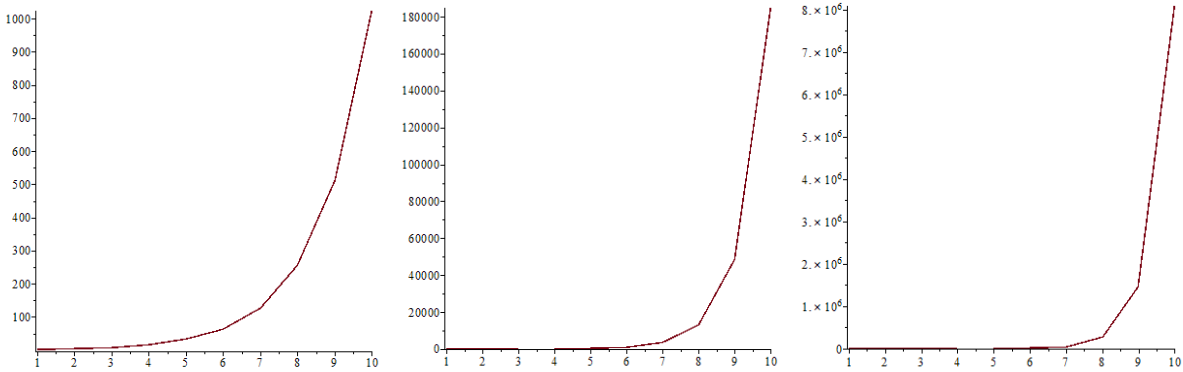
3 Exploring the Growth Constant

In this section, we will investigate the Growth Constant of each form by building the plot. And we look at the relation of growth constant between each neighbor sequences such as when $(a_1, b_1) = \{(0,1), (1,0), (1,1)\}$.

3.1 Growth Constant of the First form

Sum (binomial(n,k)*binomial(a1*n+b1*k,k),k=0..n)*x^k for all non-trivial a1, b1, x ≤ K for some fixed K;

- For $x=1$, and $(a_1, b_1) = \{(0,1), (1,0), (1,1)\}$



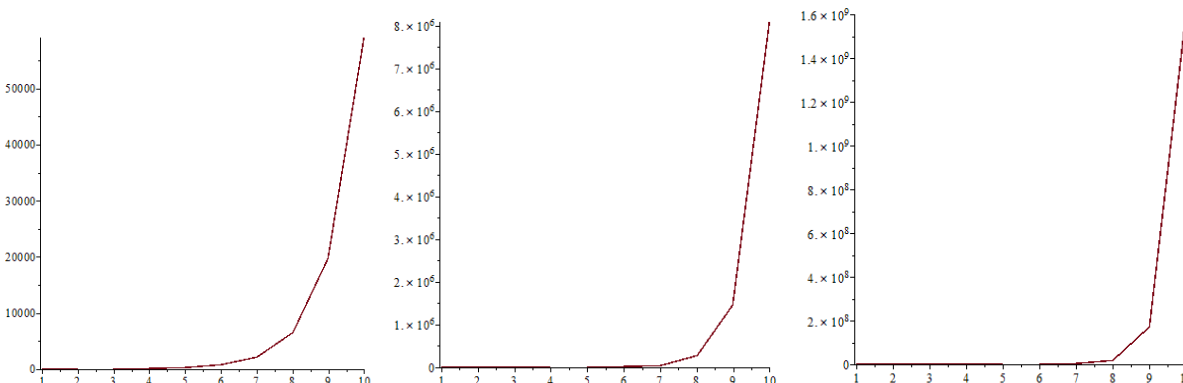
3.1.1 Plot of first 10 terms of each sequence when $x=1$

We observed that the growth of the sequence is in a 'J' shape which is similar to an exponential function. Now, define

- $A_1 = 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, \dots$
- $A_2 = 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, 2704156, 10400600, 40116600, 155117520, \dots$
- $A_3 = 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, 44642381823, \dots$

By comparing the 10th term of each sequence, $A_1[10] = 1024$, $A_2[10] = 184756$, $A_3[10] = 8097453$, $A_2[10]/A_1[10]=180.42$, $A_3[10]/A_2[10]=43.83$. And we found the growth constant of A_1 is around to 2, and the growth constant of A_2 is around to 4, and the growth constant of A_3 is around to 5.828. The growth constant is increasing by the increment of value of (a_1, b_1) , and the graph is steeper.

- For $x=2$, and $(a_1, b_1) = \{(0,1), (1,0), (1,1)\}$



3.1.2 Plot of first 10 terms of each sequence when x=2

Now, define

- $A_4 = 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441, 1594323, 4782969, 14348907, \dots$
- $A_5 = 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, 44642381823, \dots$
- $A_6 = 5, 37, 305, 2641, 23525, 213445, 1961825, 18205345, 170195525, 1600472677, 15122515985, 143457011569, 1365435096485, 13033485491077, 124715953657025, \dots$

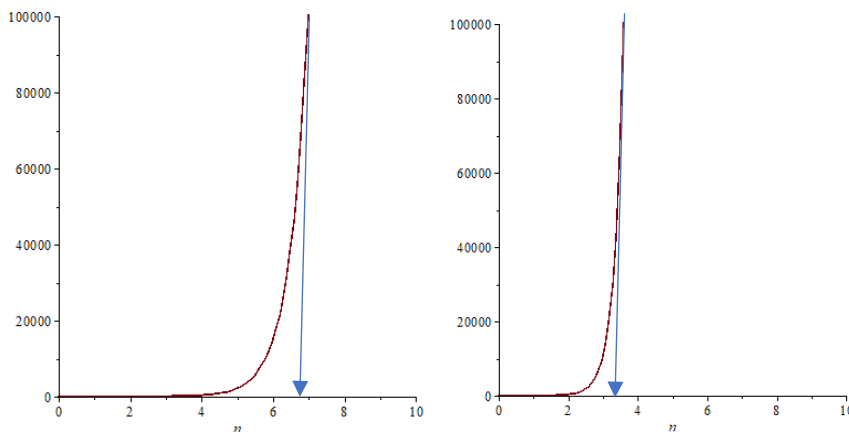
By comparing the 10th term of each sequence, $A_4[10] = 59049$, $A_2[10] = 8097453$, $A_3[10] = 1600472677$, $A_2[10]/A_1[10]=137.13$, $A_3[10]/A_2[10]=197.65$. And we found the growth constant of A_1 is around to 2, and the growth constant of A_2 is around to 4, and the growth constant of A_3 is around to 5.828. The growth constant is increasing by the increment of value of (a_1, b_1) .

With the increment of x , the ratio of corresponding N th term of each sequence is enlarging. But all the sequence is increasing in a `)` shape as we observed at first.

3.2 Growth Constant of the Second Form and Third Form

We have the same consumption about the trend of the growth constant of second form and the third form. As the input value of a_1, a_2, b_1, b_2 and x increase, the growth of the sequence is increasing faster.

We pick $(a_1, b_1, a_2, b_2) = (1, 0, 1, 0)$, $(a_1, b_1, a_2, b_2) = (1, 0, 1, 0)$ and $(a_1, b_1, a_2, b_2) = (3, 0, 3, 0)$ when x is same.



3.2.1 Comparison of the growth

The point of reaching 10^6 of two sequence is around 6.8th and 3.4th term. The value of the larger sequence has twice value of the small sequence. And we can explore that for a very large input value of a_i and b_i , the shape of growth will be more obvious.

4 Exploring the Critical Exponent and Asymptotic

In this section, we will study the graph of Critical Exponent and Asymptotic of each form. To study this section, we will use different a and b to generate the asymptotic. The asymptotic graph can give evidence to our prediction in Section 3.

4.1 Critical Exponent and Asymptotic of the First Form

For $x=1$, $(a1, b1) = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (20, 0)\}$

(a1, b1)	Critical Exponent	Asymptotic
(1, 0)	-1/2	.5641895836*4.000000000^n*(1.-.1250004001/n+0.8139071948e-2/n^2)/n^.5000000000
(2, 0)	-1/2	.4886025119*6.750000000^n*(1.-0.9722207799e-1/n+0.4726066224e-2/n^2)/n^.5000000000
(3, 0)	-1/2	.4606588660*9.481481481^n*(1.-0.9027755140e-1/n+0.4031481884e-2/n^2)/n^.5000000000
(4, 0)	-1/2	.4460310290*12.20703125^n*(1.-0.8749995647e-1/n+0.3784584933e-2/n^2)/n^.5000000000
(5, 0)	-1/2	.4370193722*14.92992000^n*(1.-0.8611152572e-1/n+0.3925278725e-2/n^2)/n^.5000000000
(20, 0)	-1/2	.4087941904*55.71925181^n*(1.-0.8353171720e-1/n+0.3967672735e-2/n^2)/n^.5000000000

For $x=1$, $(a1, b1) = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 20)\}$

(a1, b1)	Critical Exponent	Asymptotic
(0, 1)	0	1.0*2.000000000^n
(0, 2)	-1/2	.6307831306*5.000000000^n*(1.-0.6250007060e-1/n+0.4910289761e-1/n^2)/n^.5000000000
(0, 3)	-1/2	.5235457762*7.750000000^n*(1.-0.5606960745e-1/n+0.3248264413e-1/n^2)/n^.5000000000
(0, 4)	-1/2	.4843426006*10.48148148^n*(1.-0.6024859742e-1/n+0.2423611999e-1/n^2)/n^.5000000000
(0, 5)	-1/2	.4639408854*13.20703125^n*(1.-0.6394827517e-1/n+0.1884968682e-1/n^2)/n^.5000000000
(0, 20)	Cannot found	Cannot found

We found that the critical exponent is always $-1/2$ in these sequences. And the asymptotic is ordered when we increase the value of a_1 or the value of b_1 . We can find a formula for the asymptotic of the first form:

$$\text{Asymptotic} = a \cdot b^n \cdot (1-c), \text{ for } a, c \leq 1, b \geq 2$$

4.2 Critical Exponent and Asymptotic of the Second Form

For $x = 1$, $(a_1, b_1, a_2, b_2) = \{(0, 0, 1, 1), (1, 1, 0, 0), (10, 1, 0, 0), (0, 1, 2, 3), (3, 0, 3, 0)\}$

(a_1, b_1, a_2, b_2)	Critical Exponent	Asymptotic
(0, 1, 0, 1)	0	$1.0 \cdot 2.000000000^n$
(1, 0, 1, 0)	-1	$0.3675525969 \cdot 8.000000000^n \cdot (1 - 0.3333331778/n + 0.03727027504/(n^2))/n^{1.000000000}$
(1, 0, 1, 0)	-1	$0.3010076510 \cdot 44.36870803^n \cdot (1 - 0.2025500764/n + 0.05236003997/(n^2))/n^{1.000000000}$
(0, 1, 2, 3)	$-1/2$	$0.4556124674 \cdot 25.96687125^n \cdot (1 - 0.07974310136/n + 0.005813424760/(n^2))/\sqrt{n}$
(3, 0, 3, 0)	-1	$0.2198880611 \cdot 54.58757981^n \cdot (1 - 0.1915412755/n + 0.01838756636/(n^2))/n^{1.000000000}$

For $x = 2$, $(a_1, b_1, a_2, b_2) = \{(0, 0, 1, 1), (1, 1, 0, 0), (10, 1, 0, 0), (0, 1, 2, 3), (3, 0, 3, 0)\}$

(a_1, b_1, a_2, b_2)	Critical Exponent	Asymptotic
(0, 1, 0, 1)	-3	$1.0 \cdot 3.000000000^n$
(1, 0, 1, 0)	-1	$0.3724797843 \cdot 11.54196631^n \cdot (1 - 0.3273753398/n + 0.03493201969/(n^2))/n^{1.000000000}$
(1, 0, 1, 0)	-1	$0.3253074482 \cdot 77.29514157^n \cdot (1 - 0.1386612299/n + 0.1284674546/(n^2))/n^{1.000000000}$

(0, 1, 2, 3)	-1/2	$0.4644540101 \cdot 13.75072434^n \cdot (1 - 0.07295250691/n + 0.007950689567/(n^2))/\sqrt{n}$
(3, 0, 3, 0)	-1	$0.2240867806 \cdot 101.0215015^n \cdot (1 - 0.1833795288/n + 0.01903437499/(n^2))/n^{1.000000000}$

With that said, we note some similarities between the sequence presented here and the *Apery numbers* produced in Beurkes, et al. In particular, we conjecture that the asymptotic of the second form for these sequences generated by Dr. Z's algorithm is equivalent to that of the Apery numbers. Indeed, if this were the case, it would perhaps yield some interesting consequences regarding the *Riemann zeta function* in analytic number theory.

4.3 Critical Exponent and Asymptotic of the Third Form

For $x = 1$, $(a_1, b_1, a_2, b_2, a_3, b_3) = \{(1, 0, 1, 0, 1, 0), (2, 0, 2, 0, 2, 0), (10, 0, 1, 0, 2, 0), (0, 1, 1, 0, 2, 0), (3, 0, 3, 0, 3, 0)\}$

(a1,b1,a2,b2)	Critical Exponent	Asymptotic
(1, 0, 1, 0, 1, 0)	-3/2	$0.2539745438 \cdot 16.00000000^n \cdot (1 - 0.5624997294/n + 0.1271833885/(n^2))/n^{1.500000000}$
(2, 0, 2, 0, 2, 0)	-3/2	$0.1343266715 \cdot 93.65665955^n \cdot (1 - 0.3600232325/n + 0.04826458564/(n^2))/n^{1.500000000}$
(1, 0, 1, 0, 2, 0)	-3/2	$0.1684870809 \cdot 169.8131289^n \cdot (1 - 0.3485887669/n + 0.06336923974/(n^2))/n^{1.500000000}$
(0, 1, 1, 0, 2, 0)	-1	$0.3127688753 \cdot 13.03728890^n \cdot (1 - 0.2829137814/n + 0.02558783407/(n^2))/n^{1.000000000}$
(3, 0, 3, 0, 3, 0)	-3/2	$0.1074390259 \cdot 339.1485493^n \cdot (1 - 0.2846849285/n + 0.04879464340/(n^2))/n^{1.500000000}$

For $x = 2$, $(a_1, b_1, a_2, b_2, a_3, b_3) = \{(1, 0, 1, 0, 1, 0), (2, 0, 2, 0, 2, 0), (10, 0, 1, 0, 2, 0), (0, 1, 1, 0, 2, 0), (3, 0, 3, 0, 3, 0)\}$

(a1,b1,a2,b2)	Critical	Asymptotic

	Exponent	
(1, 0, 1, 0, 1, 0)	-3/2	$0.2539745438 * 16.00000000^n * (1. - 0.5624997294/n + 0.1271833885/(n^2))/n^{1.500000000}$
(2, 0, 2, 0, 2, 0)	-3/2	$0.1381356147 * 164.6242984^n * (1. - 0.3416616915/n + 0.05431748364/(n^2))/n^{1.500000000}$
(1, 0, 1, 0, 2, 0)	-3/2	$0.1828040666 * 300.8760365^n * (1. - 0.2811258950/n + 0.1233231825/(n^2))/n^{1.500000000}$
(0, 1, 1, 0, 2, 0)	-1	$0.3202931445 * 20.25000000^n * (1. - 0.2699997036/n + 0.02569646407/(n^2))/n^{1.000000000}$
(3, 0, 3, 0, 3, 0)	-3/2	$0.1101440112 * 649.3990814^n * (1. - 0.2739109733/n + 0.04687390628/(n^2))/n^{1.500000000}$

With that said, we can extend our conjecture about the asymptotic in Section 4.1 and 4.2 to the conclusion here; again, the veracity (or lack thereof) of this result would yield some interesting consequences about the *Riemann* function. Therefore, one further open question that can be investigated is the *asymptotic growth rate* of the Apery numbers.

5 Other Exploration

5.1 Other Exploration for the First Form

Add a new sequence to OEIS.org by Yifan Zhang:

A339710 "Sum (binomial (n, k) * binomial (2*n+k, k) * 2^k)"

Use the Maple function:

```
> getSequence:= proc(F,n,K,N, L1) local ope, i,Ini, ope1, L, coe1, gu, n1:
> ope:=SumTools[Hypergeometric][Zeilberger](F,n,k,N)[1];
> ope1:=expand(ope):
> L:=degree(ope1,N):
> coe1:=coeff(ope1,N,L):
> ope1:=normal(ope1/coe1):
> ope1:=normal(ope1):
> ope1:=convert([seq(factor(coeff(ope1,N,i))*N^i,i=ldegree(ope1,N)..degree(ope1,N))],`+`):
> L:=degree(ope1,N):
> Ini:=[seq(add(eval(subs({k=k1,n=n1},F)),k1=0..n1),n1=1..degree(ope,N))]:
```

```

>
> if nops(Ini)<>L then
> ERROR('Ini should be of length`, L):
> fi:
> ope1:=expand(subs(n=n-L,ope1)/N^L):
> gu:=Ini:
>
> for n1 from nops(Ini)+1 to L1 do
> gu:=[op(gu), -add(gu[nops(gu)+1-j1]*subs(n=n1,coeff(ope1,N,-j1)),
> j1=1..L)]:
> od:
>
> end:

```

- We can find as many sequences as in the form of $\sum (\binom{n}{k} \binom{a_1 n + b_1 k}{k}, k=0..n) x^k$ for all non-trivial $a_1, b_1, x \leq K$ for some fixed K ;

5.2 Analysis of Growth Trends

We consider Dutch combinatorial number theorist Fritz Beurkes' paper titled "Some Congruences for the Apery Numbers". One of the paper's foremost strengths is that it revisits and makes significant progress in proving some powerful results about the Apery sequence of numbers; More particularly, the author demonstrates a *stronger* result that generalizes those of previous works in the Journal of Number Theory.

PROPOSITION. *Let $c_1, c_2, c_3, \dots, \in \mathbb{Z}, c_1 = 1$. Then*

$$c_{mp^r} \equiv c_{mp^{r-1}} \pmod{p^r} \quad \forall m, r \in \mathbb{N}, \forall \text{ primes } p \Leftrightarrow \exp\left(\sum_1^{\infty} \frac{c_n}{n} T^n\right) \in \mathbb{Z}[[T]].$$

Thus, for example, the congruences in Theorem 2 imply that

$$\exp\left(\sum_1^{\infty} \frac{a_{n-1}}{n} T^n\right) \in \mathbb{Z}[[T]].$$

While our contribution as students in Math 454 is to use Maple to investigate this result in combinatorics and number theory, we also supply a description of the following general trend in the academic practices used to study combinatorics and finite mathematical structures.

Even though the proposition is number-theoretic, i.e. *enumerative* of a certain class of integers that satisfies the given property, the proof uses several important properties from algebraic geometry. We see this as interesting because, in the class, we studied the enumerative properties of a combinatorial structure completely separately from the algebraic properties of the same; hence, the *correspondence* discovered by Beurkes, et. al., between the two has served to broaden our knowledge and perspectives about combinatorics and the related fields of math in a myriad of unexpected ways.