Title: Analysis of Sequences Enumerating 2D Lattice Walks

Introduction:

This project serves as a final project for the class Combinatorics run by Prof. Doron Zeilberger during the Fall 2020 semester at Rutgers University. In this project, we construct a database of sequences enumerating 2D lattice walks for various sets of atomic steps. Furthermore, we analyze various characteristics of these sequences including their recurrences, growth rates, critical exponents, asymptotics, and congruence properties. Finally, we utilize Maple to programmatically look at various random paths for different sets of atomic steps and determine the statistical properties of these paths.

Context:

The classical case often studied in many introductory combinatorics classes is enumerating the number of walks from a point (0,0) to a point (n,n) for some integer n >= 1 where the only steps that can be made are (0,1) and (1,0). From analyzing this example, it can be seen that the number of such walks can be calculated using binomial coefficients. Another interesting case that is often looked at are the walks such that x >= y at all points in the walk which are also known as Dyck paths. The number of these paths can be determined by the nth Catalan number. Although these cases have been well studied and the sequences enumerating walks for this set of atomic steps are well known and can be found within the OEIS, there are numerous other sets of atomic steps that provide interesting details to analyze. As such, we have documented a list of such sets of atomic steps and produced sequences enumerating the number of walks for each of them. Moreover, we have produced the sequences for the number of walks where x >= y at all points in the walk which we will call “good walks” throughout the rest of the analysis.

Methodology:

Since there are an infinite number of sets of atomic steps, we had to determine which ones would be the most interesting to look at. Some inspiration for these came from looking at the movement patterns of various chess pieces. Others came from using steps containing numbers of mathematical importance like prime numbers or powers of two.

We added functions to randomly generate walks and good walks and specific functions to analyze the random walks such as checking the average number of repeated steps, good walks, and visits to the diagonal for a given number of repeat runs. We also created a function to visualize the generated random walks.

For every set of atomic steps we looked at, we utilized the InfoA and InfoGA procedures in order to find the corresponding data we needed. We produced 50 terms for each sequence and calculated the estimated asymptotics to order 2 for each set of steps.

Once we had compiled all of our data into organized and ordered files, we searched through the data looking for general trends and patterns across all of the parameters we were looking at.

Results/Analysis:

ANALYZING SEQUENCE PROPERTIES:

One feature we noticed was that we were unable to determine recurrence operators and other characteristics for sequences that had 0s in them. Such sequences included the sequence derived from the chess knight and many of the sequences involving families of [x^2,0] and [2^x,0]. Additionally, when scouring sequences of subsets of {[i,j] : i = 0..2, j = 0..2} - {[0,0]}, many of them were unable to produce useful and interesting sequences because no walks were able to made for certain values of n. Generally, subsets including [0,1] and [1,0] were more interesting to look at since those steps ensured that there would always be a walk from (0,0) to (n,n).

For recurrence operators, it seems that every sequence that we could find a recurrence operator were of order 4 or less. Sequences corresponding to larger sets of atomic steps tended to have operators of higher orders. For growth constants, one interesting trait we found was that for any given set of atomic steps, the growth constant for the sequence of regular walks is always equivalent to the growth constant for the sequence of good walks.

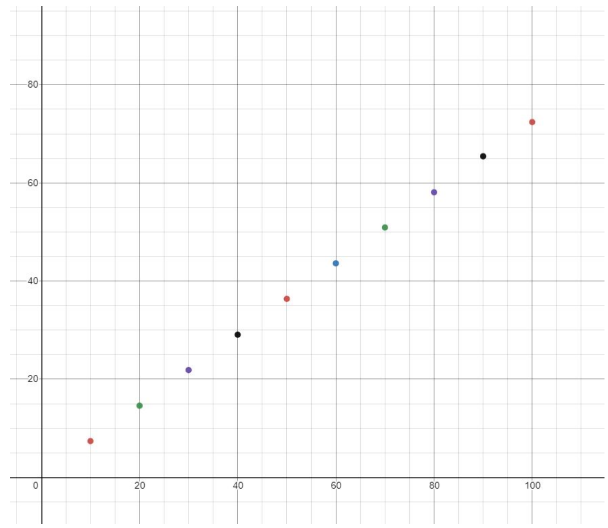
In terms of congruence properties, we focused on finding out whether each of the sequences at the ith prime number modulo the ith prime resulted in the same number for various ranges of i such that it could fit our data. We found that none of the sequences involving families of steps [x^2, 0] or [2^x,0] contained any congruence properties. Meanwhile, most of the sequences found from looking at chess pieces except the Knight resulted in having some form of congruence property. Additionally, while most sequences that had congruence properties tended to have L[ithprime(i)] mod ithprime(i) = 1, the sequences involving chess pieces had L[ithprime(i)] mod ithprime(i) > 1. As a general trend, it seems like sequences derived from larger sets of atomic steps are more likely to have congruence properties with higher numbers.

For critical exponents, there were two common patterns. The first pattern is where the sequence of regular walk's critical exponent is -1/2 and its corresponding sequence of good walk's critical exponent is -3/2, having a difference of 1. This seems to occur when there is a large variation in growth of the two. Otherwise, the critical exponents of regular walks and good walks are equivalent. For estimated asymptotics, we set k=2. For most of the sequences, the estimated asymptotic equation reached the order of 2. However, for some of the simpler sets with smaller atomic steps, the order only reached 1. An example is the {1,1} 1D reduction and the walk of the Chess Bishop.

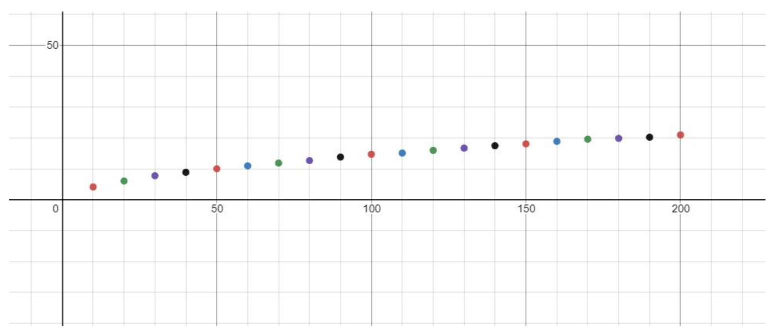
After finding all of the sequences we wanted for our database, we cross-referenced them with the OEIS to see which sequences were already in there and which were new. Many of the sequences derived from movement of chess pieces were found in the OEIS as well as those derived from small subsets of {[i,j] : i = 0..2, j = 0..2} - {[0,0]}. Several of the sequences we found were not in the OEIS and so, we submitted the sequence derived from atomic steps {(0,1),(1,0),(1,1),(1,2),(2,1)} to the OEIS which can now be found as A339565.

ANALYZING STATISTICAL PROPERTIES:

- {[1,1],[2,2]}
    - The average visits to the diagonal was [7.37, 14.591, 21.849, 29.069, 36.371, 43.606, 50.912, 58.091, 65.428, 72.402] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].
    - **GENERAL TREND**: The number of direction changes seems to linearly scale.

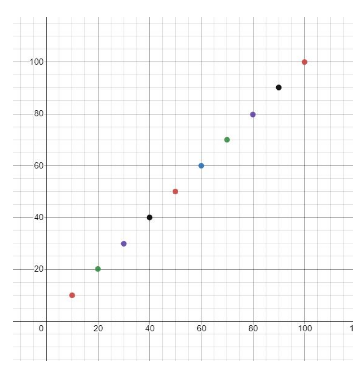


- {[1, 1], [1, 2], [2, 1], [2, 2]}
    - The average visits to the diagonal was [4.15, 6.063, 7.786, 8.887, 10.037, 10.969, 11.875, 12.69, 13.82, 14.728,15.112, 15.989, 16.728, 17.497, 18.082, 18.896, 19.596, 19.883, 20.253, 20.984] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].



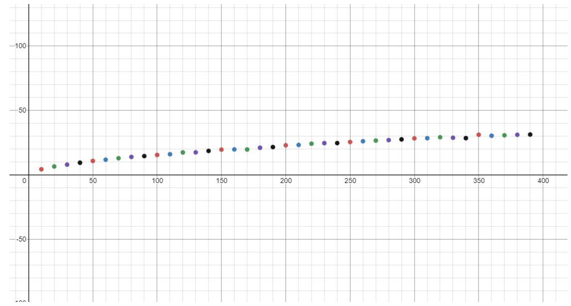
    - **GENERAL TREND**: The number of direction changes seems to logarithmically scale.

- {[0,1],[1,0]}
    - The average visits to the diagonal was [9.953, 20.101, 29.839, 39.97, 50.032, 59.971, 69.998, 79.746,90.186, 100.02] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].



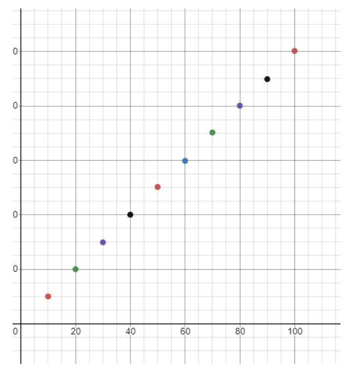
- **GENERAL TREND**: The number of direction changes seems to linearly scale.

- {[1, 1], [1, 2], [2, 0], [2, 1], [2, 2]}
    - The average number of visits to the diagonal was [4.311, 6.493, 7.93, 9.398, 10.779, 11.73, 12.884, 13.921, 14.548, 15.446] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i]. The growth is quite slow. The number of visits to the diagonal from [0,0] to [1000,1000] is 50.149.

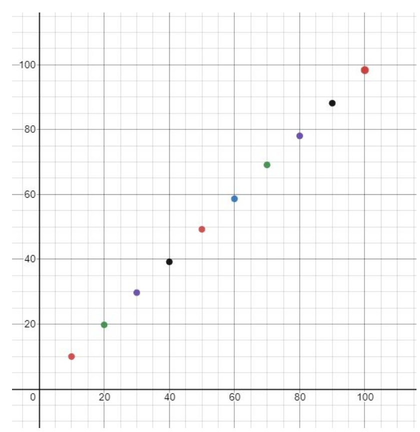


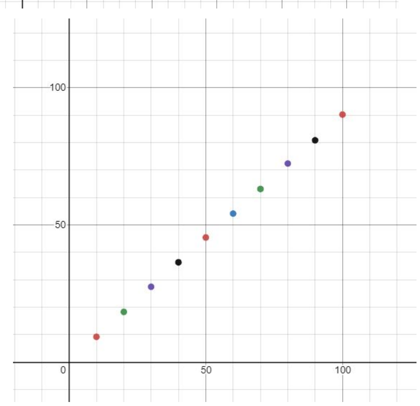
    - **GENERAL TREND:** The number of direction changes seems to logarithmically scale.

- {[x^4,0],[0,1]}
    - The average number of direction changes was [9.983, 19.987, 29.863, 40.027, 50.217, 59.823, 70.257, 80.103, 89.857, 100.228] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].
    - **GENERAL TREND:** The number of direction changes seems to linearly scale.
    - average proportion of repeat steps: 0.49005
    - average proportion of good walks: 0.5638641176470588235294117647058823529412



- {[x^3,0],[0,1]}
    - The average number of direction changes was [10.016, 19.785, 29.729, 39.218, 49.246, 58.661, 69.114, 78.075, 88.159, 98.344]] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].
    - **GENERAL TREND:** The number of direction changes seems to linearly scale.
    - average proportion of repeat steps:
    - 0.4890667216804201050262565641410352588147
    - average proportion of good walks:
    - 0.5588032058014503625906476619154788697174

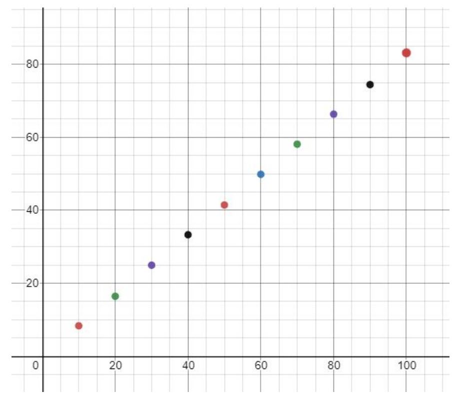




- {[x^2,0],[0,1]} x squared
    - The number of direction changes was [9.188, 18.27, 27.435, 36.333, 45.389, 54.106, 63.065, 72.339, 80.838,

90.193] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].
    - **GENERAL TREND:** The number of direction changes seems to linearly scale.
    - average number of repeat steps:
    - 0.46887962277671190982230160469846663992 65
    - average proportion of good walks:
    - 0.53878249881653794070612584641023815635 27

- {[i,0],[0,1]}
    - The number of direction changes was [8.352,16.431, 24.96,33.304, 41.468, 49.858, 58.119,66.356,74.452, 83.152] where the i-th index is the number of direction changes for lattice walks going from [0,0] to [10*i,10*i].
    - **GENERAL TREND:** The number of direction changes seems to linearly scale.
    - average number of repeat steps:
    - 0.42575388873059494286141063330859867626 56
    - average proportion of good walks:
    - 0.56086575187610974178769284586680412532 12



The **general trend** amongst steps with the form of {[x^n,0],[0,1]}, the average number of direction changes will scale linearly. The average proportion of repeat steps seems to slowly increase from 0.42, and the average proportion of good walks seems to stay around the same value of mid 0.5's.

A lot of the average number of visits to the diagonal tended to be logarithmically growing. So, after going from [0,0] to [i,i], the growth would dramatically decrease after i=20 for many of them.

The **general trend** for the average proportion of repeat steps tended to hover around 1/(# of given steps). For example, if there were 5 steps given, the proportion was around 0.2. The **general trend** for the average proportion of good walks generally hovered between 0.4 and 0.6 even in cases where the steps were heavily skewed in one direction (for example, {[x^2,0],[0,1]}) unless the steps were only ones that were in the form [x,x]. The proportion seemed to be on the higher end of the 0.4-0.6 range when the steps were more close together, such as {[1, 2], [2, 1], [2, 2]} as opposed to {[x^3,0],[0,1]}.