Solutions to Real Quiz 10 of Dr. Z.'s Dynamical Models in Biology class

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1. By hand solve the system

$$\frac{dx}{dt} = -x - y \quad , \frac{dy}{dt} = x + y \quad , \quad x(0) = 1 \quad , \quad y(0) = 1 \quad .$$

Plot, by hand, the phase-plane diagram.

Solution to 1: The system is

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad ,$$

where A is the 2×2 matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \quad .$$

Let's look for the eigenvalues

$$\det \begin{bmatrix} -1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (-1-\lambda)(1-\lambda) + 1 = \lambda^2 \quad .$$

Setting $\lambda^2 = 0$, we get a **double root** $\lambda = 0$.

Since $e^{0 \cdot t} = 1$ the general solution is

$$\mathbf{x}(t) = \mathbf{a} + \mathbf{b}t \quad ,$$

for some vectors $\mathbf{a}$ and $\mathbf{b}$.

Since $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we have

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad .$$

Differentiating with respect to t we get

$$\mathbf{x}'(t) = \mathbf{b} \quad ,$$

Plugging-in $t = 0$ we get

$$\mathbf{x}'(0) = \mathbf{b} \quad ,$$

But

$$\mathbf{b} = \mathbf{x}'(0) = A\mathbf{x}(0) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad ,$$

so

$$\mathbf{b} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Hence the solution in vector form is

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Spelling it out

$$x(t) = 1 - 2t \quad , \quad y(t) = 1 + 2t \quad .$$

The phase-plane diagram is simply a straight line with slope -1 crossing the y-axis at $(2, 0)$ and the x-axis at $(0, 2)$. The arrow of time is pointing in the Northeast direction from $(1, 1)$.

Comment: A much easier way would be to 'cheat' and convert it to a scalar diff. eq. as follows

Adding the two equations we have

$$\frac{d(x+y)}{dt} = 0 \quad .$$

So $x + y$ is a **constant** and using the initial conditions we get $x + y = 2$ so

$$y = 2 - x \quad .$$

Plugging into the first equation we have

$$x'(t) = -2 \quad ,$$

so $x(t) = -2t+c$ for some constant. But $x(0) = 1$ so $c = 1$ and $x(t) = -2t+1$ and $y(t) = 2-x(t) = 1 + 2t$, much faster without linear algebra.