

hw 9

hw 9 Zachariah Ali

1. $a(n+2) = 5a(n+1) - 6a(n) = 0$

$a(0) = 1, a(1) = 2$
Assuming it's meant to be

Suppose $a(n) = z^n$ is a solution.

$$z^{n+2} = 5z^{n+1} - 6z^n$$

$$z^{n+2} - 5z^{n+1} + 6z^n = 0$$

$$z^2 - 5z + 6 = 0$$

$$\Rightarrow z_1 = 3, z_2 = 2$$

So $a(n) = 3^n \times A + 2^n \times B$.

$$a(0) = A + B = 1 \Rightarrow A = 1 - B$$

$$a(1) = A(3) + B(2) = 2$$

$$\Rightarrow (1-B)(3) + B(2) = 2$$

$$(3) - B(3) + B(2) = 2$$

$$B(2-3) = 2-3$$

$$B = \frac{2-3}{2-3}$$

$$A = 1 - \frac{2-3}{2-3} = \frac{2-3}{2-3}$$

$$\text{So } a(n) = \frac{2-3}{2-3} (3)^n + \left(\frac{2-3}{2-3} \right) (2)^n$$

$$Z = \frac{700}{1000} = 0.7$$

$$(i) \frac{700}{10,000} = 700(1000 - 300)$$

$$(ii) 6.13 \times 10^{-6} = \frac{1 - \left(\frac{0.51}{0.44}\right)^{700}}{1 - \left(\frac{0.51}{0.44}\right)^{1000}}$$

$$(iii) 0.2881 = \frac{1 - \left(\frac{0.501}{0.441}\right)^{700}}{1 - \left(\frac{0.501}{0.441}\right)^{1000}}$$

3. Let $a(n)$ be the probability you ~~win~~ exit a winner.
We have prob. p that you advance with ~~prob~~ to $n+1$ dollar.
Similarly $1-p$ that you regress to $n-1$.

$$So \quad a(n) = pa(n+1) + (1-p)a(n-1)$$

with boundary conditions $a(0) = 0, a(L) = 1$.

$$\text{Standard form: } a(n+2) = \frac{a(n+1) + (p-1)a(n)}{p}$$

$$= \frac{1}{p}a(n+1) + \left(1 - \frac{1}{p}\right)a(n)$$

$$\text{Suppose } a(n) = z^n.$$

$$z^{n+2} = \frac{1}{p}z^{n+1} + \left(1 - \frac{1}{p}\right)z^n$$

$$z^2 = \frac{1}{p}z + \left(1 - \frac{1}{p}\right)$$

$$z^2 - \frac{1}{p}z + \frac{1}{p} - 1 = 0$$

$$z_{1,2} = \frac{\frac{1}{p} \pm \sqrt{\frac{1}{p^2} - 4 \times \left(\frac{1}{p} - 1\right)}}{2 \times 1}$$

$$\frac{1}{p^2} = 4\left(\frac{1}{p} - 1\right)$$

$$= \frac{1}{p^2} + 4 - \frac{4}{p}$$

$$= \left(\frac{1}{p} - 2\right)^2$$

$$So \quad z_{1,2} = \frac{2}{2} = 1 \quad \text{or} \quad z_2 = \frac{1}{p} - 1$$

$$So \text{ the general solution is } a(n) = A \times 1^n + B\left(\frac{1}{p} - 1\right)^n$$

A, B are constants.

$$\text{We have } a(0) = 0, a(L) = 1.$$

So TPO

$$a(n) = A + \left(\frac{1}{p} - 1\right)^n B$$

$$a(0) = A + B = 0 \Rightarrow A = -B$$

$$a(L) = A + \left(\frac{1}{p} - 1\right)^L B = 1$$

$$\Rightarrow -B + \left(\frac{1}{p} - 1\right)^L B = 1$$

$$\Rightarrow B = \frac{1}{-1 + \left(\frac{1}{p} - 1\right)^L}$$

$$\Rightarrow A = \frac{-1}{-1 + \left(\frac{1}{p} - 1\right)^L}$$

$$\text{So } a(n) = \frac{-1 + \left(\frac{1}{p} - 1\right)^n}{-1 + \left(\frac{1}{p} - 1\right)^L}$$

But at $p = \frac{1}{2}$ $\left(\frac{1}{p} - 1\right) = 1$ so B is undefined.

Let's analyse this. We have $p = \frac{1}{2}$. $a(n) = A + Bn$
Characteristic at $p = \frac{1}{2}$. $z^2 - 2z + 1 = 0$

$$z_1 = 1, z_2 = 1.$$

$$\text{So } a(n) = A + Bn.$$

$$a(0) = 0.$$

$$a(0) = A + B \times 0 = A = 0.$$

$$a(L) = 1$$

$$a(L) = BL = 1$$

$$B = \frac{1}{L}.$$

$$a(n) = \frac{n}{L}.$$

$$\text{So } a(n) = \begin{cases} \frac{1 - \left(\frac{1}{p} - 1\right)^n}{1 - \left(\frac{1}{p} - 1\right)^L}, & \text{if } p \neq \frac{1}{2} \\ \frac{n}{L}, & \text{if } p = \frac{1}{2} \end{cases}$$

4. ~~Let~~ Let $b(n)$ be the expected number of rounds you will play if you have n dollars. We have prob $\frac{1}{2}$ that you have $n+1$ dollars i.e. $b(n+1)$ rounds to go. Similarly for $b(n-1)$. We have also then played another round.

$$\text{So } b(n) = \frac{1}{2} b(n+1) + \frac{1}{2} b(n-1) + 1.$$

$$\text{Standard form } b(n+1) = 2b(n) - b(n-1) - 2.$$

We already have from Q3 that the homogeneous solution is $b_h(n) = A + Bn$.

Now we note that $b(0) = 0$ and $b(L) = 0$ as at 0 or L dollars you are done.

For particular solution suppose $b_p(n) = Kn^2$ as we had a double root in homogeneous workings and we have nonhomogeneous constant term.

$$\text{So } Kn^2 = \frac{1}{2} K(n+1)^2 + \frac{1}{2} K(n-1)^2 + 1$$

$$Kn^2 = \frac{K}{2}(n^2 + 2n + 1) + \frac{K}{2}(n^2 - 2n + 1) + 1$$

$$Kn^2 = Kn^2 + K + 1$$

$$\text{So comparing coefficients } 0 = K + 1 \Rightarrow K = -1.$$

$$\text{General solution } b(n) = A + Bn - n^2$$

$$b(0) = A = 0$$

$$b(L) = BL - L^2 = 0$$

$$\Rightarrow L(B - L) = 0$$

$$\Rightarrow B = L$$

$$\text{So } b(n) = n(L - n)$$