Homework for Lecture 9 of Dr. Z.'s Dynamical Models in Biology class

Version of Oct. 5, 2025 (Correcting a typo, found by Rachel Adelmam, who won a dollar).

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 6, 2025.

Subject: hw9

with an attachment hw9FirstLast.pdf

1. Solve the boundary value problem

$$a(n+2) \;=\; 5\,a(n+1) \;-\; 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

1. Solve the boundary value problem

$$a(n+2) \;=\; 5\,a(n+1) \;-\; 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

<u>Sol</u> :

$$r^2 - 5r - 6 = 0$$

$$(r-2)(r-3) = 0.$$

$$\text{so, } r = 2, \; r = 3.$$

$$a(n) = A\cdot 2^n + B\cdot 3^n \qquad \text{with} \quad a(0) = 1, \; a(L) = 2.$$

$$a(n) = A\cdot 2^n + B\cdot 3^n = 1 \qquad \text{---} \; ①$$

$$a(L) = A\cdot 2^L + B\cdot 3^L = 2. \qquad \text{--} \; ②$$

$$n=0, \; A+B = 1$$

$$n=1 \quad 2A+3B = 1 \quad \Rightarrow 2A = 1-3B \quad \text{so, } A = \frac{1-3B}{2}$$

subsitute into ②, we get

$$\frac{1-3B}{2}\cdot 2^L + B\cdot 3L = 2$$

$$(1-3B)\cdot 2^{L-1} + B\cdot 3^L = 2$$

$$2^{L-1} - 3\cdot B\cdot 2^{L-1} + B\cdot 3^L = 2$$

$$B(3^L - 3\cdot 2^{L-1}) = 2 - 2^{L-1}$$

$$B = \frac{2-2^{L-1}}{3^L - 3\cdot 2^{L-1}}$$

$$\text{Then : } A = \frac{1-3B}{2} = \frac{1 - 3\cdot\frac{2-2^{L-1}}{3^L - 3\cdot 2^{L-1}}}{2}$$

$$= \frac{3^L - 6}{2\cdot(3^L - 3\cdot 2^{L-1})}$$

$$\text{so, } a(n) = \frac{3^L - 6}{3^L - 3\cdot 2^{L-1}}\cdot 2^{n-1} + \frac{2-2^L}{3^L - 3^{L-1}}\cdot 3^n$$

for any $L > 1$.

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

Sol : $p = 0.5$,

1) prob. winning : $P_{win} = \frac{n}{L} = \frac{700}{1000} = 0.7$

a) Expected duration :

$Z_n = n(L-n) = 700(1000-700) = 210000$ rounds.

2) $p = 0.49$, $q = 0.51$, $r = q/p = 0.51/0.49 > 1$.

$P_{win} = \frac{r^n - 1}{r^L - 1} = \frac{r^{700}-1}{r^{1000}-1}$ here $r \approx 1.0408\ldots$

$\approx 6.14 \times 10^{-6}$

so, the probility of winning is about 6.14×10^{-6}.

3). $p = 0.499$, $q = 0.501$, $r = 0.501/0.499 > 1$

$P_{win} = \frac{r^n - 1}{r^L - 1} = \frac{r^{700}-1}{r^{1000}-1}$ here $r \approx 1.004008\ldots$

≈ 0.2876

so, the probility of winning is about 0.2876.

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

Sol. Let P_n be probility of reaching L before 0 starting with n.

one step analysis:

$$P_n = p \cdot P_{n+1} + q P_{n-1} \quad . \quad P(0) = 0, \quad P(L) = 1$$

Let $P_n = r^n$,

$$r^n = p \cdot r^{n+1} + q r^{n-1}$$

divide by r^{n-1} on both sides,

$$r = p r^2 + q$$

$$p r^2 - r + q = 0$$

$$r = \frac{1 \pm \sqrt{1-4pq}}{2p}$$

since $1 - 4pq = 1 - 4p(1-p) = 1 - 4p + 4p^2$
$= (2p-1)^2$

so $r = \dfrac{1 \pm |2p-1|}{2p}$

Case 1: $p \neq q \Rightarrow 2p - 1 \neq 0$

so, $r_1 = 1$, $r_2 = q/p$

So $P_n = A + B(q/p)^n$

$P(0) = A + B = 0 \Rightarrow A = -B$

$P(L) = A + B(q/p)^L = 1 \Rightarrow -B + B(q/p)^L = 1$

$B((q/p)^L - 1) = 1 \Rightarrow B = \dfrac{1}{(q/p)^L - 1}$

so, $P(n) = -B + B \cdot (q/p)^L = B((q/p)^L - 1) = \dfrac{(q/p)^n - 1}{(q/p)^L - 1}$

Let $r = q/p$, then smiplify

$$P_n = \frac{r^n - 1}{r^L - 1}$$

Case 2. $p = q = 1/2$.

Then $P(n) = A + Bn$ $\quad P(0) = 0$, $P(L) = 1$.

$$\Rightarrow \begin{cases} A = 0 \\ A + BL = 1 \end{cases} \Rightarrow B = \frac{1}{L}.$$

$$P(n) = \frac{1}{L} \cdot n = \frac{n}{L}$$

$$\text{Thus } P(n) = \begin{cases} \frac{n}{L}, & \text{if } p = q = 1/2 \\ \frac{(q/p)^n - 1}{(q/p)^L - 1} & \text{if } p \neq q \end{cases}$$

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L - n)$.

Pf: Let Z_n = expected number of rounds starting with n untill exit.

one-step-anylysis:

$$Z_n = 1 + \frac{1}{2} Z_{n+1} + \frac{1}{2} Z_{n-1} \qquad Z_0 = 0, \; Z_L = 0$$

Solve for Homogenous eqn.

$$Z_{n+1} - 2Z_n + Z_{n-1} = -2.$$

Assume $(Z_n)_h = A + Bn$.

particular soultion is $(Z_n)_p = Cn^2$.

$$C(n+1)^2 - 2 \cdot C \cdot n^2 + C(n-1)^2 = C(n^2 + 2n + 1 - 2n^2 + n^2 - 2n + 1)$$
$$= 2C = -2$$

$$\Rightarrow C = -1.$$

So, $Z_n = A + Bn - n^2$

$Z(0) = 0$, $Z(L) = 0$

$$\Rightarrow \begin{cases} A = 0 \\ BL - L^2 = 0 \end{cases} \Rightarrow \left. \begin{aligned} L(B-L) &= 0 \\ L &\neq 0 \end{aligned} \right\} \Rightarrow B = L.$$

thus, $Z_n = Ln - n^2 = n(L-n)$.

Therefore, statement proved ☒