Dynamical Models in Biology — Homework 9

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1. **Boundary-value problem for a linear recurrence.**

   Solve
   $$a(n+2) = 5\,a(n+1) - 6\,a(n), \qquad a(0) = 1, \quad a(L) = 2.$$
   Characteristic equation $r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0$, so
   $$a(n) = A \cdot 2^n + B \cdot 3^n.$$
   Use the boundary data:
   $$A + B = 1, \qquad A \cdot 2^L + B \cdot 3^L = 2.$$
   From $A = 1 - B$,
   $$(1-B)2^L + B3^L = 2 \;\Rightarrow\; B = \frac{2 - 2^L}{3^L - 2^L}, \qquad A = 1 - B = \frac{3^L - 2}{3^L - 2^L}.$$
   Hence the solution (valid for any integer n and given $L \geq 1$) is
   $$\boxed{a(n) = \frac{3^L - 2}{3^L - 2^L}\,2^n \;+\; \frac{2 - 2^L}{3^L - 2^L}\,3^n.}$$

2. **Gambler's ruin: $i = 700$, $L = 1000$.**

   Let p be the probability of winning \$1 at each step (and $q = 1 - p$). The probability of exiting a winner (hitting L before 0), starting at i, is
   $$\Pr[\text{win}] = \begin{cases} \dfrac{i}{L}, & p = \frac{1}{2}, \\ \dfrac{1 - (q/p)^{\,i}}{1 - (q/p)^{\,L}}, & p \neq \frac{1}{2}. \end{cases}$$

   (i) For $p = \frac{1}{2}$:
   $$\Pr[\text{win}] = \frac{700}{1000} = 0.7.$$

   (i') Expected number of rounds until absorption (fair case).
   For $p = \frac{1}{2}$, the expected duration is
   $$\mathbb{E}[T] = i(L - i) = 700 \cdot 300 = 210{,}000 \text{ rounds}.$$

   (ii) For $p = 0.49$ ($q = 0.51$, $r = q/p = 0.51/0.49$):
   $$\Pr[\text{win}] = \frac{1 - r^{700}}{1 - r^{1000}} \approx 6.134 \times 10^{-6}.$$

(iii) For $p = 0.499$ ($q = 0.501$, $r = 0.501/0.499$):

$$\Pr[\text{win}] = \frac{1 - r^{700}}{1 - r^{1000}} \approx 0.288156.$$

3. **Derive the general winning-probability formula.**

   Let $u_n = \Pr[\text{hit } L \text{ before } 0 \mid X_0 = n]$ for a nearest-neighbor walk on $\{0, 1, \ldots, L\}$ with up-probability p and down-probability $q = 1 - p$. Then for $1 \le n \le L - 1$,

   $$u_n = p\, u_{n+1} + q\, u_{n-1}, \qquad u_0 = 0, \quad u_L = 1.$$

   For $p \neq q$, the general solution to $p\, u_{n+1} - u_n + q\, u_{n-1} = 0$ is $u_n = C_1 + C_2\, r^{\,n}$ with $r = \frac{q}{p}$. Enforcing the boundary conditions,

   $$u_n = \frac{1 - r^{\,n}}{1 - r^{\,L}}.$$

   For $p = \frac{1}{2}$ (so $r = 1$) we take the continuous limit and get the linear solution $u_n = \dfrac{n}{L}$.

4. **Expected duration in a fair casino.**

   Let $T_n = \mathbb{E}[\text{time to hit } 0 \text{ or } L \mid X_0 = n]$ with $p = q = \frac{1}{2}$. The first-step recurrence is

   $$T_n = 1 + \tfrac{1}{2}\, T_{n+1} + \tfrac{1}{2}\, T_{n-1}, \qquad T_0 = T_L = 0.$$

   Rearrange to $T_{n+1} - 2T_n + T_{n-1} = -2$. Guess a quadratic $T_n = an^2 + bn + c$:

   $$\bigl(a(n+1)^2 + b(n+1) + c\bigr) - 2(an^2 + bn + c) + \bigl(a(n-1)^2 + b(n-1) + c\bigr) = -2,$$

   which simplifies to $2a = -2 \Rightarrow a = -1$. Using $T_0 = 0$ gives $c = 0$, and $T_L = 0$ gives $-L^2 + bL = 0 \Rightarrow b = L$. Thus

   $$\boxed{T_n = n(L - n).}$$

   This matches the value used in 2(i′) for $n = 700$, $L = 1000$.