Homework for Lecture 9 of Dr. Z.'s Dynamical Models in Biology class

Version of Oct. 5, 2025 (Correcting a typo, found by Rachel Adelmam, who won a dollar).

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 6, 2025.

Subject: hw9

with an attachment hw9FirstLast.pdf

1. Solve the boundary value problem

$$a(n+2) \,=\, 5\,a(n+1) \,-\, 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

1. Solve the boundary value problem

$$a(n+2) \;=\; 5\,a(n+1) \;-\; 6\,a(n) = 0 \quad ; \quad \overset{a(0)=0}{\cancel{a(n) = 1}}, \quad a(L) = 2 \quad .$$

$$z^2 = 5z - 6$$

$$z^2 - 5z + 6 = 0$$

$$(z-3)(z-2) = 0$$

$$z = 2, 3$$

$$a(n) = A_1 (2)^n + A_2 (3)^n$$

$$a(0) = 0 = A_1 + A_2 \rightarrow A_1 = -A_2$$

$$a(L) = 2 = A_1 2^L + A_2 3^L$$

$$2 = A_1 2^L - A_1 3^L$$

$$2 = A_1 (2^L - 3^L)$$

$$A_1 = \frac{2}{2^L - 3^L}$$

$$A_2 = \frac{-2}{2^L - 3^L}$$

$$a(n) = \frac{2}{2^L - 3^L} (2)^n - \frac{2}{2^L - 3^L} (3)^n$$

$$\boxed{a(n) = \frac{2(2^n - 3^n)}{2^L - 3^L}}$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5) $\frac{n}{L}$

- have \$700
- max = \$1000

$\frac{n}{L} = \frac{700}{1000} = \boxed{.7}$

(i') What is the expected number of rounds until you exit either a winner or loser?

$a(n) = \frac{1}{2}a(n+1) + \frac{1}{2}a(n-1)$, $a(0)=0$, $a(L)=1$

$z^2 - 2z + 1 = 0$

$a(n) = A_1 n + A_2$

$0 = A_2$, $1 = A_1 \cdot L$

$A_1 = 1/L$

$n(L-n)$

$b(n) = \frac{1}{2}b(n+1) + \frac{1}{2}b(n-1) + 1$

$b(L) = 0$ $\qquad$ $b(L) = 0$

$b(n) = n(L-n)$

$= 700(1000-700) = \boxed{210{,}000 \text{ rounds}}$

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

ii) p = probability you win \$1

$q = 1-p$ = prob you lose \$1

$$a(n) = p\,a(n+1) + (1-p)\,a(n-1)$$

$$pz^2 - z + (1-p) = 0$$

$$(z-1)(pz + p - 1) = 0$$

$$z_1 = 1 \qquad z_p = \frac{1-p}{p} = \frac{q}{p}$$

$$a(n) = A_1 + A_2\left(\frac{q}{p}\right)^n$$

$$0 = A_1 + A_2 \qquad A_1 + A_2\left(\frac{q}{p}\right)^L = 1$$

$$A_1 = \left(1 - \left(\frac{p}{q}\right)^L\right) = 1$$

$$A_1 = \frac{1}{1-\left(\frac{p}{q}\right)^L} = -A_2$$

$$a(n) = \frac{1-(q/p)^n}{1-(q/p)^L} = \frac{1-\left(\frac{.51}{.49}\right)^{700}}{1-\left(\frac{.51}{.49}\right)^{1000}} = \boxed{6.134\times10^{-6}}$$

iii)

$$\frac{1-\left(\frac{.501}{.499}\right)^{700}}{1-\left(\frac{.501}{.499}\right)^{1000}} = \boxed{.288}$$

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

p = probability you win \$1

$q = 1 - p$ = prob you lose \$1

$$a(n) = pa(n+1) + (1-p)a(n-1)$$

$$pz^2 - z + (1-p) = 0$$

$$(z-1)(pz+p-1) = 0$$

$$z_1 = 1 \qquad z_p = \frac{1-p}{p} = \frac{q}{p}$$

$$a(n) = A_1 + A_2\left(\frac{q}{p}\right)^n$$

$$0 = A_1 + A_2 \qquad A_1 + A_2\left(\frac{q}{p}\right)^L = 1$$

$$A_1 = \left(1 - \left(\frac{p}{q}\right)^L\right) = 1$$

$$A_1 = \frac{1}{1 - \left(\frac{p}{q}\right)^L} = -A_2$$

$$a(n) = \frac{1 - (q/p)^n}{1 - (q/p)^L}$$

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

$$b(n) = pb(n+1) + qb(n-1) + 1$$
$$b(n) = \frac{1}{2}b(n+1) + \frac{1}{2}b(n-1) + 1$$
$$b(n+1) - 2b(n) + b(n-1) = -2$$

$b(n+1) \to$ if won
$b(n-1) \to$ if lost
$b(0) = b(L) = 0$

$$b(n+1) - 2b(n) + b(n-1) = 0$$
$$b(n+2) - 2b(n+1) + b(n) = 0$$
$$z^2 - 2z + 1 = 0$$
$(z-1)^2 \to$ double root
$z = 1, 1$

homog side

$$b(n) = c_1(1)^n + c_2(n)(1)^n$$
$$\hookrightarrow b(n) = c_1 + c_2 n$$

nonhomog guess

$$b(n) = an^2$$
$$a(n+1)^2 - 2an^2 + a(n-1)^2 = -2$$
$$an^2 + 2an + a + 2an^2 + an^2 - 2an + a = -2$$
$$2a = -2$$
$$a = -1$$

$$b(n) = c_1 + c_2 n - n^2$$

$b(0) = 0$
$\hookrightarrow 0 = c_1$
$b(L) = 0 = c_2 L - L^2$
$c_2 = L$

$$b(n) = Ln - n^2$$
$$\boxed{b(n) = n(L-n)}$$