Homework 9: DMB Ezra Chechik

1) BVP: $a(n+2) = 5a(n+1) - 6a(n) = 0$;

$a(0) = 1, a(L) = 2.$

$\rightarrow z^2 - 5z + 6 = 0$

$(z-2)(z-3) = 0 \quad , z_1 = 2 \quad , z_2 = 3$

so, $a(n) = A z_1^n + B z_2^n$

$1 = A + B \quad ; \quad A \cdot 2^L + B \cdot 3^L = 2$

$\rightarrow B = 1 - A \rightarrow A \cdot 2^L + (1-A) \cdot 3^L = 2$

$A \cdot 2^L + 3^L - A3^L = 2 \rightarrow A(2^L - 3^L) + 3^L = 2$

$\rightarrow A = \dfrac{2-3^L}{2^L - 3^L}, B = \dfrac{2^L - 3^L}{2^L \cdot 3^L} - \dfrac{2-3^L}{2^L \cdot 3^L} = \dfrac{2^L - 2}{2^L - 3^L}$

$$\boxed{a_n = \frac{2-3^L}{2^L-3^L} \cdot 2^n + \frac{2^L-2}{2^L-3^L} \cdot 3^n}$$

2) Currently: 700 $\rightarrow$ Max amount: 1000

i) when probability $p = 0.5 \rightarrow a(n) = \dfrac{n}{L} = \dfrac{700}{1000} = 0.7$

i') In Fair game $n(L-n)$ = expected rounds

$\rightarrow 700(1000-700) = 210{,}000$

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ii) $p = 0.49$, $q = 0.51$

$$a(n) = \frac{1 - (q/p)^n}{1 - (q/p)^L}$$

$$\rightarrow a(700) = \frac{1 - (0.51/0.49)^{700}}{1 - (0.51/0.49)^{1000}}$$

$\rightarrow a(700) = 2 \times 10^{-5}$ (basically 0% chance)

iii) $p = 0.499$, $q = 0.501$

$$a(700) = \frac{1 - (0.501/0.499)^{700}}{1 - (0.501/0.499)^{1000}}$$

$a(700) = 0.29$ (approx.) (Better chance of winning)

$\boxed{3.}$ $a(n) \rightarrow$ prob you exit winner given n current dollars.

Recurrence (Boundary Problem):

$a(n) = p\,a(n+1) + q\,a(n-1)$, $a(0) = 0$, $a(L) = 1$

$$pz^2 - z + q = 0 \rightarrow (z-1)(pz + p - 1) = 0$$

$z_1 = 1$, $z_2 = \frac{1-p}{p} = \frac{q}{p}$

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$$a(n)_{general} = A_1 + A_2 (q/p)^n$$

$$0 = A_1 + A_2 \; ; \; 1 = A_1 + A_2 (q/p)^L$$

$$\boxed{A_1 = -A_2} \qquad 1 = -A_2 + A_2 (q/p)^L$$

$$1 = A_2\left((q/p)^L - 1\right), \; A_2 = \frac{1}{(q/p)^L - 1}$$

$$\text{so, } a(n) = \frac{-1}{(q/p)^L - 1} + \frac{1}{(q/p)^L - 1}(q/p)^n$$

$$\text{simplify} \rightarrow \underline{a(n)}_{\underline{win}} = \frac{1 - (q/p)^n}{1 - (q/p)^L}$$

4) prove expected duration in a Fair Casino is $n(L-n)$

$$c(n) = \frac{1}{2}[c(n+1) + c(n-1)] + 1$$

$$b(0) = 0 \quad , \quad b(L) = 0$$

$$\rightarrow c(n+1) - 2c(n) + c(n-1) = -2$$

$$z^2 - 2z + 1 = 0 \rightarrow (z-1)^2 = 0$$

$$C_{gen}(n) = An + B \quad \xrightarrow[\text{page}]{\text{next}}$$

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$C_{partic}(n) = \alpha n^2 \rightarrow \alpha[(n+1)^2 - 2n^2 + (n-1)^2] = -2$

$\alpha 2 = -2 \quad , \alpha = -1$

$C(n) = An + B - n^2$

$\rightarrow 0 = B \quad \& \quad AL - L^2 = 0 \quad , A = L$

SO, $C(n) = Ln - n^2 = n(L-n)$ ✓