1. Solve the boundary value problem

$$a(n+2) = 5\,a(n+1) - 6\,a(n) = 0 \quad ; \quad \overset{a(0)=0}{\cancel{a(n)=1}}, \quad a(L) = 2 \quad .$$

$a(n+2) = 5a(n+1) - 6a(n) = 0$

$z^2 = 5z - 6$

$z^2 - 5z + 6 = 0$

$(z-3)(z-2) = 0$

$\left.\begin{matrix} z_1 = 3 \\ z_2 = 2 \end{matrix}\right\} a(n) = A_1(3)^n + A_2(2)^n$

$\begin{cases} a(0) = 0 = A_1 + A_2 \Rightarrow A_2 = -A_1 \\ a(L) = 2 = A_1(3)^L + A_2(2)^L \end{cases}$

$2 = A_1(3)^L - A_1(2)^L$

$2 = A_1(3^L - 2^L)$

$\left.\begin{matrix} A_1 = \frac{2}{3^L - 2^L} \\ A_2 = -\frac{2}{3^L - 2^L} \end{matrix}\right\} \boxed{a(n) = \frac{2}{3^L - 2^L}(3)^n - \frac{2}{3^L - 2^L}(2)^n}$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

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> read(`DMB10.txt`)
  For a list of the Main procedures type: Help10(); for help with a specific procedure type: Help10(ProcedureName); for example Help10(RandomMC);
  For a list of the other procedures type: Help10a(); for help with a specific procedure type: Help10(ProcedureName); for example Help10(RandomMC);
> evalf(ExactPD(700, 1000, 1/2))
                    [0.7000000000, 210000.]
> evalf(ExactPD(700, 1000, 49/100))
                    [6.134387118 × 10^-6, 34999.69328]
> evalf(ExactPD(700, 1000, 499/1000))
                    [0.2881559204, 205922.0398]
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i) 0.7 is the prob

i') 210,000 rounds

ii) 6.134×10^{-6} is the prob

iii) 0.28815 is the prob

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

p = you win \$1

$q = 1-p$ (you lose a \$1)

$a(n) = pa(n+1) + qa(n-1)$

$a(n) = pa(n+1) + (1-p)a(n-1)$

$pz^2 - z + (1-p) = 0$

$(z-1)(pz-q) = 0$

$\left.\begin{matrix} z_1 = 1 \\ z_2 = \frac{q}{p} \end{matrix}\right\} a(n) = A_1 + A_2\left(\frac{q}{p}\right)^n$

$\begin{cases} a(0) = 0 = A_1 + A_2 \; ; \; A_1 = -A_2 \\ a(L) = 1 = A_1 + A_2\left(\frac{q}{p}\right)^L \end{cases}$

$1 = A_1 - A_1\left(\frac{q}{p}\right)^L$

$1 = A_1\left(1 - \left(\frac{q}{p}\right)^L\right)$

$\left.\begin{matrix} A_1 = \frac{1}{1-(q/p)^L} \\ A_2 = -\frac{1}{1-(q/p)^L} \end{matrix}\right\} \boxed{a(n) = \frac{1}{1-(q/p)^L} - \frac{1}{1-(q/p)^L}\left(\frac{q}{p}\right)^n}$

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

win : $b(n+1)$

lose : $b(n-1)$

$b(0) > b(L) = 0$

$$b(n) = p\,b(n+1) + q\,b(n-1) + 1$$
$$b(n) = \frac{1}{2} b(n+1) + \frac{1}{2} b(n-1) + 1$$
$$b(n+1) - 2b(n) + b(n-1) = -2$$

$$b(n+1) - 2b(n) + b(n-1) = 0$$
$$b(n+2) - 2b(n+1) + b(n) = 0$$
$$z^2 - 2z + 1 = 0$$
$(z-1)(z-1) = 0$ double root

$z = 1 \rightarrow a(n) = C_1(1)^n + C_2 n (1)^n$

↳ homog: $a(n) = C_1 + C_2 n$

↳ non-homog: $b(n) = an^2$

$$a(n+1)^2 - 2an^2 + a(n-1)^2 = -2$$
$$an^2 + 2n + a + 2an^2 + an^2 - 2an + a = -2$$
$$2a = -2$$
$$a = -1$$

$$b(n) = C_1 + C_2 n - n^2$$
$$b(0) = 0 = C_1$$
$$b(L) = 0 = C_2 L - L^2$$
$$C_2 = L$$

$b(n) = Ln - n^2$

$\boxed{b(n) = n(L-n)}$