Homework for Lecture 9 of Dr. Z.'s Dynamical Models in Biology class

Version of Oct. 5, 2025 (Correcting a typo, found by Rachel Adelmam, who won a dollar).

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 6, 2025.

Subject: hw9

with an attachment hw9FirstLast.pdf

1. Solve the boundary value problem

$r^2 - 5r + 6 = 0$
$(r-2)(r-3) \to r_1 = 2 , r_2 = 3$
$a_0 = A2^n + B3^n = 1$
$a_L = A2^L + B3^L = 2$

$A = \frac{2-3^L}{2^L-3^L} 2^n$ $\quad B = \frac{2^L-2}{2^L-3^L}$ $\quad a_n = \frac{2-3^L}{2^L-3^L} 2^n + \frac{2^L-2}{2^L-3^L} 3^n$

$$a(n+2) = 5\,a(n+1) - 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

$\frac{n}{L} = \frac{700}{1000} = .7$

(i') What is the expected number of rounds until you exit either a winner or loser? $n(L-n) = 700(300) = 210000$

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ? $\frac{1-(q/p)^n}{1-(q/p)^L} \approx 6.134 \times 10^{-6}$

$p = .49 \quad q = .51$

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ? $\frac{1-(q/p)^n}{1-(q/p)^L} \approx .288$

$p = .499 \quad q = .501$

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

3) $a(n) = pa(n+1) + qa(n+1)$

$pz^2 - z + q = 0$
$(z-1)(pz + p - 1)$
$z_1 = 1 \quad z_2 = q/p$
$a(n) = A_1 + A_2 (q/p)^n$
$A_1 + A_2 = 0 \Rightarrow A_2 = -A_1$
$A_1 + A_2 (q/p)^L = 1$
$\hookrightarrow A_1 (1 - (q/p)^L) = 1$
$A_1 = \frac{1}{1-(q/p)^L}$
$a_n = \frac{1-(q/p)^n}{1-(q/p)^L}$

4) $b(n) = 1 + \frac{1}{2} b(n+1) + \frac{1}{2} b(n-1) \quad b(0) = 0 \quad b(L) = 0$

$b(n+1) + b(n-1) = (n+1)(L-n-1) + (n-1)(L-n+1) = 2n(L-n) - 2$

$1 + \frac{1}{2}[b(n+1) + b(n-1)] = 1 + \frac{1}{2}[2n(L-n) - 2]$
$= n(L-n) = b(n)$