Homework for Lecture 9 of Dr. Z.'s Dynamical Models in Biology class

Version of Oct. 5, 2025 (Correcting a typo, found by Rachel Adelmam, who won a dollar).

Email the answers (as a .pdf file) to

ShaloshBEkhad@gmail.com

by 8:00pm Monday, Oct. 6, 2025.

Subject: hw9

with an attachment hw9FirstLast.pdf

1. Solve the boundary value problem

$$a(n+2) \;=\; 5\,a(n+1) \;-\; 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

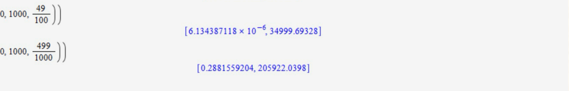
(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.



1. Solve the boundary value problem

$$a(n+2) = 5\,a(n+1) - 6\,a(n) = 0 \quad ; \quad \overset{a(0)=0}{\cancel{a(n)=1,}} \quad a(L) = 2 \quad .$$

$z^2 - 5z + 6 = 0$
$(z-2)(z-3) = 0$
$z = 2, 3$

$a(n) = C_1 2^n + C_2 3^n$

$a(0) = 0 = C_1 + C_2$
$C_1 = -C_2$

$a(L) = 2 = C_1 2^L + C_2 3^L$
$2 = -C_2 2^L + C_2 3^L$

$\frac{2}{3^L - 2^L} = C_2$

$\frac{-2}{3^L - 2^L} = C_1$

$a(n) = \frac{-2}{3^L - 2^L}(2)^n + \frac{2}{3^L - 2^L}(3)^n$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

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> read(`DMB10.txt`)
      For a list of the Main procedures type: Help10(); for help with a specific procedure type: Help10(ProcedureName); for example Help10(RandomMC);
      For a list of the other procedures type: Help10a(); for help with a specific procedure type: Help10(ProcedureName); for example Help10(RandomMC);
> evalf(ExactPD(700, 1000, 1/2))
                                   [0.7000000000, 210000.]
> evalf(ExactPD(700, 1000, 49/100))
                                [6.134387118 × 10^-6, 34999.69328]
> evalf(ExactPD(700, 1000, 499/1000))
                                  [0.2881559204, 205922.0398]
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(i) Prob. $= 0.7$

210000 rounds expected

(ii) Prob $= 6.134 \times 10^{-6}$

(iii) Prob $= 0.288$

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

$a(0) = 0$
$a(L) = 1$

$p = \text{win } \$1 \qquad q = \text{lose } \$1 = 1 - p$

$a(n) = p\,a(n+1) + q\,a(n-1)$

$a(n+1) = p\,a(n+2) + q\,a(n)$
$a(n+2) = \frac{1}{p} a(n+1) - \frac{q}{p} a(n)$

$z^2 - \frac{1}{p} z + \frac{1-p}{p} = 0$
$(z-1)\left(z - \frac{1-p}{p}\right)$

$a(n) = \frac{-1}{\frac{q}{p} - 1} + \frac{1}{\frac{q}{p} - 1}\left(\frac{q}{p}\right)^n$

$a(n) = C_1 + C_2\left(\frac{q}{p}\right)^n$

$a(0) = 0 = C_1 + C_2$
$C_1 = -C_2$

$a(L) = 1 = C_1 + C_2 \frac{q}{p}$
$1 = \frac{q}{p} C_2 - C_2$

$\frac{1}{\frac{q}{p} - 1} = C_2$

$\frac{-1}{\frac{q}{p} - 1} = C_1$

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

$b(n) = pb(n+1) + qb(n-1) + 1$

$b(n) = \frac{1}{2} b(n+1) + \frac{1}{2} b(n-1) + 1$

$b(n+1) - 2b(n) + b(n-1) = -2$

$b(n+1) \to \text{win}$

$b(n-1) \to \text{lose}$

$b(0) = b(L) = 0$

$b(n+1) - 2b(n) + b(n-1) = 0$

$b(n+2) - 2b(n+1) + b(n) = 0$

$z^2 - 2z + 1 = 0$

$(z-1)(z-1)$

$b(n) = C_1 + C_2 n$

nonhomog: $b(n) = an^2$

$a(n+1)^2 - 2an^2 + a(n-1)^2 = -2$

$an^2 + 2n + a + 2an^2 + an^2 - 2an + a = -2$

$2a = -2$

$a = -1$

$b(n) = C_1 + C_2 n - n^2$

$b(0) = 0 = C_1$

$b(L) = 0 = C_2 L - L^2$

$L = C_2$

$b(n) = Ln - n^2$