Homework for Lecture 9 of Dr. Z.'s Dynamical Models in Biology class

Version of Oct. 5, 2025 (Correcting a typo, found by Rachel Adelmam, who won a dollar).

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 6, 2025.

Subject: hw9

with an attachment hw9FirstLast.pdf

1. Solve the boundary value problem

$$a(n+2) \;=\; 5\,a(n+1) \;-\; 6\,a(n) = 0 \quad ; \quad a(n) = 1, \quad a(L) = 2 \quad .$$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

3. Derive, all by yourself, the formula for the prob. of exiting a winner in a gambler's ruin where the prob. of winning a dollar is p if the max. amount if L and you currently have n dollars.

4. Prove that the expected duration of staying in a fair casino with max. amount if L and you currently have n dollars is $n(L-n)$.

Daniyal Chawdhry HW9

→ Not sure if there was a typo in the question but I will solve this

① $a(n+2) = 5a(n+1) - 6a(n)$, $\underbrace{a(n) = 1}_{}$, $a(L) = 2$

I will also consider this to be $a(0) = 1$?

$z^2 - 5z + 6 : z = 2, 3$

$$\left.\begin{aligned} A_1 + A_2 &= 1 \\ A_1 (2)^L + A_2 (3)^L &= 2 \end{aligned}\right]$$

$A_1 (2)^L + (1 - A_1)(3)^L = 2$

$A_1 (2)^L - A_1 (3)^L = 2 - (3)^L$

$$a(n) = \left(\frac{2 - (3)^L}{(2)^L - (3)^L}\right)(2)^n + \left(1 - \frac{2 - 3^L}{2^L - 3^L}\right)(3)^n \Longleftarrow A_1 = \frac{2 - (3)^L}{(2)^L - (3)^L} \quad A_2 = 1 - A_1$$

$$\boxed{a(n) = 3^n + (2^n - 3^n)\left(\frac{2 - 3^L}{2^L - 3^L}\right)}$$

② i) $\frac{7}{10}$ ii) 210,000 iii) 6.13×10^{-6} iv) 0.288

③ We know $a(n) = q\, a(n-1) + p\, a(n+1)$ where $a(0) = 0$, $a(L) = 1$

$a(n+2) = \frac{1}{p} a(n+1) - \frac{q}{p} a(n)$

$z^2 - \frac{1}{p} z + \frac{1-p}{p} = 0 \Rightarrow (z - 1)\left(z - \frac{1-p}{p}\right)$.

$$\left.\begin{aligned} A_1 + A_2 &= 0 \\ A_1 + A_2 \left(\tfrac{1-p}{p}\right)^L &= 1 \end{aligned}\right\} \quad A_1 - A_1 \left(\tfrac{1-p}{p}\right)^L = 1$$

$$A_1 = \frac{1}{1 - \left(\frac{1-p}{p}\right)^L} \quad A_2 = -A_1$$

$$a(n) = \left(\frac{-1}{1 - \left(\frac{1-p}{p}\right)^L}\right)\left(\frac{(1-p)^n}{p}\right) + \left(\frac{1}{1 - \left(\frac{1-p}{p}\right)^L}\right) = \boxed{\frac{1 - \left(\frac{1-p}{p}\right)^n}{1 - \left(\frac{1-p}{p}\right)^L}}$$

④ $b(n) = \frac{1}{2} b(n-1) + \frac{1}{2} b(n+1) + 1$ $b(0) = 0$ $b(L) = 0$

$b(n+1) - 2b(n) + b(n-1) = -2$

nonhomogeneous:

① y_h: $z^2 - 2z + 1 = 0$

$(z-1)(z-1) \Rightarrow A_1 + A_2 n$

② y_p: Try αn^2 as αn or α won't work.

$\hookrightarrow \alpha(n+1)^2 - \underline{2\alpha(n^2)} + \alpha(n-1)^2 = -2$

$\alpha\,(\underline{2n - 2n} + 1 + 1) = -2 \Rightarrow \alpha = -1$

③ $y = A_1 + A_2 n - n^2$

where $b(0) = 0$ $b(L) = 0$

$A_1 = 0$ $\cancel{A_1} + A_2 L - L^2 = 0$

$A_2 = L$

$$\boxed{y = Ln - n^2 = \boxed{n(L-n)}}$$