Dyn Mod in Bio HW9 Rachel Adelman

1. Solve the boundary value problem

$$a(n+2) = 5\,a(n+1) - 6\,a(n) \quad a(0)=0 \quad a(L)=2$$

$z^2 = 5z - 6$

$z^2 - 5z + 6 = 0$

$(z-2)(z-3) = 0$

$z = \{2, 3\}$

gen sol'n: $a(n) = C_1 2^n + C_2 3^n$

$a(0) = 0 = C_1 2^0 + C_2 3^0 \qquad C_1 = -C_2$

$a(L) = 2 = C_1 2^L + C_2 3^L$

$2 = -C_2 2^L + C_2 3^L$

$2 = C_2(-2^L + 3^L)$

$\dfrac{2}{-2^L + 3^L} = C_2$

$C_1 = -\left[\dfrac{2}{-2^L+3^L}\right] \Rightarrow \dfrac{-2}{2^L - 3^L}$

$a(n) = \left[\dfrac{-2}{2^L-3^L}\right] 2^n + \dfrac{(-1)}{(-1)}\left[\dfrac{2}{-2^L+3^L}\right] 3^n$

Sol'n: $\boxed{a(n) = \left[\dfrac{2}{2^L-3^L}\right](2^n - 3^n)}$

2. (Use a calculator or Maple) In a gambler's ruin problem you currently have 700 dollars and the maximum amount is 1000.

(i) What is the probability of exiting a winner if your prob. of winning a dollar is 0.5 (and losing a dollar is 0.5)

(i') What is the expected number of rounds until you exit either a winner or loser?

(ii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.49 ?

(iii) What is the prob. of exiting a winner if your prob. of winning a dollar is 0.499 ?

i) $P_{win} = \dfrac{n}{L} = \dfrac{700}{1000} = 0.7$

(favorable)

i') duration: $b(n) = n(L-n)$

$= 700(1000-700)$

$= 700(300)$

$= 210000$

maple code. read `DMB10.txt.'

input: ExactPD(700, 1000, $\frac{1}{2}$)

output: [$\frac{7}{10}$, 210,000]

$\boxed{\text{prob of win/loose } 0.7 \quad \text{duration: } 210000 \text{ rounds}}$

ii) $p = 0.49 \Rightarrow$ not favorable

must use $a(n) = \dfrac{1-\left(\frac{q}{p}\right)^n}{1-\left(\frac{q}{p}\right)^L}$ where, $q = 1-p$

$$P_{win} \approx \frac{1-\left(\frac{1-0.49}{0.49}\right)^{700}}{1-\left(\frac{1-0.49}{0.49}\right)^{1000}}$$

maple: ExactPD(700, 1000, $\frac{49}{100}$)

$[6.13438 \times 10^{-6}, 34999.69]$

$\boxed{\text{Pwin} = 6.13438E\text{-}6 \quad \text{Duration: } 34999.69 \text{ rounds}}$

$b(n) = d_1 - d_2$

$$d_1 = \frac{n}{1-2p} \Rightarrow = \frac{700}{1-2(0.49)} = 35{,}000$$

$$d_2 = \frac{L}{1-2p}(P_{win}) = \frac{1000}{1-2(0.49)}(6.134E-6)$$

$b(n) = 35{,}000 - 0.3067$
$= 34999.69$

iii) $p = 0.499 \Rightarrow$ not favorable, but closer to 0.5

$P_{win} = a(n) = \dfrac{1-\left(\frac{q}{p}\right)^n}{1-\left(\frac{q}{p}\right)^L}$ where, $q = 1-p$

$$P_{win} = \frac{1-\left(\frac{1-0.499}{0.499}\right)^{700}}{1-\left(\frac{1-0.499}{0.499}\right)^{1000}}$$

$P_{win} = 0.2882$

duration:

$b(n) = d_1 - d_2$

$$d_1 = \frac{n}{1-2p} \Rightarrow \frac{700}{1-2(0.499)} = 350000$$

$$d_2 = \frac{L}{1-2p}(P_{win}) \Rightarrow \frac{1000}{1-2(0.499)}(0.2882)$$
$= 144100$

$b(n) = 350000 - 144077$

$= 205900$

maple: ExactPD(700, 100, $\frac{499}{1000}$)

[0.2882, 205922]

$\boxed{\begin{array}{l} P(\text{win}) = 0.2882 \\ \text{Duration: } 205922 \text{ rounds} \end{array}}$

3. let $a(n)$ be prob. of reaching capital L (exit a winner) starting w/ n dollars. On each round you gain $+1$ w/ a prob. of p and -1 w/ a prob. of $1-p$. The boundary cond. are $a(0)=0$ and $a(L)=1$

$0 \le n \le L-2$

Recurrence eqn: $a(n) = p\,a(n+1) + (1-p)a(n-1)$

put in standard form: $a(n+1) = (1-p)a(n) + p\,a(n+2)$

$p\,a(n+2) = a(n+1) - (1-p)a(n)$

$a(n+2) = \frac{1}{p}a(n+1) - \left(\frac{1-p}{p}\right)a(n)$

$z^2 - \frac{1}{p}z + \frac{1-p}{p} = 0$

$(z-1)(z-\frac{1-p}{p}) = 0$

$z = \{1, \frac{1-p}{p}\}$

$a(n) = A_1(1)^n + A_2\left(\frac{1-p}{p}\right)^n$

$\hookrightarrow a(n) = A_1 + A_2\left(\frac{1-p}{p}\right)^n$

plug in initial cond.

$a(0) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$a(L) = A_1 + A_2\left(\frac{1-p}{p}\right)^L = 1$

$-A_2 + A_2\left(\frac{1-p}{p}\right)^L = 1$

$A_2\left(\left(\frac{1-p}{p}\right)^L - 1\right) = 1$

$\frac{1}{2} \cdot \frac{2}{1}$

$A_2 = \frac{1}{\left(\frac{1-p}{p}\right)^L - 1}$ $\quad A_1 = \frac{-1}{\left(\frac{1-p}{p}\right)^L - 1}$

sol'n:

$a(n) = \frac{-1}{\left(\frac{1-p}{p}\right)^L - 1} + \frac{1}{\left(\frac{1-p}{p}\right)^L - 1}\left(\frac{1-p}{p}\right)^n$

simplify

$\hookrightarrow a(n) = \frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^L - 1} \cdot \frac{(-1)}{(-1)}$

$\hookrightarrow \boxed{a(n) = \frac{1-\left(\frac{1-p}{p}\right)^n}{1-\left(\frac{1-p}{p}\right)^L}}$

Special case: when $p = 1/2$ $\quad \frac{1-p}{p} = \frac{1-\frac{1}{2}}{\frac{1}{2}} = 1$

put in standard form:

$a(n+1) = \frac{1}{2}a(n) + \frac{1}{2}a(n+2)$

$-\frac{1}{2}a(n+2) = -a(n+1) + \frac{1}{2}a(n)$

$a(n+2) = 2a(n+1) - a(n)$

$z^2 = 2z - 1$

$z^2 - 2z + 1 = 0$

$z = \{1, 1\}$ double root!

$a(n) = A_1 z_1^n + A_2 n z_1^n$

$\hookrightarrow z_1 = 1$

$a(n) = A_1(1)^n + A_2 n(1)^n$

gen solution $\rightarrow a(n) = A_1 + A_2 n$

$a(0) = 0 = A_1 \rightarrow A_1 = 0$

$a(L) = 1 = A_1 + A_2 L \rightarrow A_2 L = 1$

$A_2 = \frac{1}{L}$

$a(n) = 0 \cdot (1)^n + \left(\frac{1}{L}\right)n$

sol'n: $\boxed{a(n) = \frac{n}{L}}$

4. let $b(n)$ be the expected # of rounds until 0 or L is reached, starting w/ n dollars, when $p = 1/2$

BC: $b(0) = 0$, $b(L) = 1$

for $1 \leq n \leq L-1$

$-2\left(b(n) = 1 + \frac{1}{2}b(n+1) + \frac{1}{2}b(n-1)\right)$

$-2b(n) = -2 - b(n+1) - b(n-1)$

$-2b(n) + b(n+1) + b(n-1) = -2$

$-2b(n) = \qquad = 2a$

comparing the two

$2a = -2 \Rightarrow a = -1$

$b(n) = an^2 + bn + c$

$b(n) = -n^2 + bn + c$

sol'n: $b(n) = -n^2 + Ln = n(L-n)$ ✓

$b(0) = 0 = c \Rightarrow c = 0$

$b(L) = -L^2 + bL = 0$

$L = b$