1.)

```
r1 := raRF(x, 3);

                    2 x^3 + 6 x^2  + 10 x + 7
               r1 := ----------------------
                      x ^3+ 5 x^2 + 6 x + 4

SSS((2*x^3 + 6*x^2 + 10*x + 7)/(x^3 + 5*x^2 + 6*x + 4), x);
        [      /  4       3                    \]
        [RootOf\_Z^4  + 3 _Z^2  - 6 _Z - 7, index = 1/]

Roots: X = -2.484
         X = 1.529

r2 := raRF(x, 3);

                    4 x ^3 + 2 x^2  + 2 x + 10
               r2 := ----------------------

                    10 x^3  + 9 x^2  + 3 x + 8

SSS(r2, x);
    [      /     4       3     2                    \]
    [RootOf\10 _Z  + 5 _Z  + _Z  + 6 _Z - 10, index = 1/]
          Roots: X = -1.274
                 X = 0.74


r3 := raRF(x, 3);

                    9 x^3 + 10 x^2  + 8 x + 2
               r3 := ----------------------
                    7 x^3  + 7 x^2  + 6 x + 1

SSS(r3, x);
    [RootOf\ 7Z^4  - 2 Z^3  - 4 Z^2  - 7 Z - 2, index = 1/]
          Roots: X = -0.325
                 X = 1.36
r4 := raRF(x, 3);

                    8 x^3  + 3 x^2 + 5x + 3
               r4 := -------------------------------
                     x^3 + x^2  + 3x + 10
```

```
SSS(r4, x);
[(2116 + 12*sqrt(2649))^(1/3)/6 + 80/(3*(2116 + 12*sqrt(2649))^(1/3)) + 8/3] = 6.126


r5 := raRF(x, 3);

                    10 x^3  + 5 x^2  + 5 x + 3
               r5 := ----------------------------------
                    8 x^3  + 2 x^2  + 10 x + 10

SSS(r5, x);
    [RootOf\   8Z^4  - 8 Z ^3 + 5 Z ^2 + 5 Z - 3, index = 1/]
Roots: X = -0.674
          X = 0.468

r6 := raRF(x, 3);

                     5 x^3  + 9 x^2  + 2 x + 4
                r6 := -------------------------------
                     3 x^3  + 4 x^2  + 7 x + 9
SSS(r6, x);
     [RootOf\  3 Z^4  - Z^3  - 2 Z^2  + 7Z - 4, index = 1)]
Roots: X = -1.513
          X = 0.655

r7 := raRF(x, 3);

                    4 x^3  + 10 x^2  + 9 x + 10
               r7 := ----------------------------------
                    9 x^3  + 8 x^2 + 7 x + 9
SSS(r7, x);
                         [1]


r8 := raRF(x, 3);

                     4 x^3 + 2 x^2  + 10 x + 1
                r8 := ----------------------
                      6 x^3 + x^2  + 8 x + 7

SSS(r8, x);
    [RootOf\  6Z^4  - 3 Z^3  + 6Z^2  - 3Z - 1, index = 1/]
Roots: X = -0.221
          X = 0.671
```

r9 := raRF(x, 3);

$$r9 := \frac{2x^3 + 7x^2 + x + 4}{9x^3 + 9x^2 + 5x + 9}$$

SSS(r9, x);
[RootOf\ 9Z^4 + 7Z^3 - 2Z^2 + 8Z - 4, index = 1/]
Roots: X = -1.475
X = 0.435

r10 := raRF(x, 3);

$$r10 := \frac{6x^3 + 4x^2 + 3x + 4}{8x^3 + 10x^2 + 4x + 4}$$

SSS(r10, x);
[RootOf\ 8Z^4 + 4Z^3 + Z - 4, index = 1/]
Roots: X = -1.048
X = 0.7

2.) Find the terms between 2000-th and 2030-th terms of the orbit,
starting at x0 = 0.5 of the recurrence xn = kx(n−1)(1 − x(n−1)),
for k = 1, k = 2, k = 2.5, k = 3.1, k = 3.2, k = 3.3, k = 3.5. What do you find?

For k =1,
Orb(f, x, 0.5, 2000, 2030)
= [0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005, 0.0005,
0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049,
0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049, 0.00049]

All terms in this orbit are approaching 0.

For k =2,
Orb(f, x, 0.5, 2000, 2030)
= [0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5,
0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5,]

Since 0.5 is a steady state point, for any xi in this orbit, xi = 0.5

For k =2.5,

Orb(f, x, 0.5, 2000, 2030)
= [0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6]

For this f(x), it has a steady stable point at 0.6. By the 2000th iteration, it has already reached 0.6. Thus, all iterations after x(i) for i>2000 will be 0.6.

For k =3.1,
Orb(f, x, 0.5, 2000, 2030)
= [0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765, 0.558, 0.765]

It appears that there is an unstable steady-state point in between (0.558, 0.765), causing the output of the function to bounce between those two values.

For k =3.2,
Orb(f, x, 0.5, 2000, 2030)
= [0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799, 0.513, 0.799]

It appears that there is an unstable steady-state point between (0.513, 0.799), causing the output of the function to oscillate between those two values. It is also worth noting that these values are very similar to the function when k = 3.2.

For k =3.3,
Orb(f, x, 0.5, 2000, 2030)
= [0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824, 0.479, 0.824]

It appears that there is an unstable steady-state point between (0.479, 0.824), causing the output of the function to oscillate between these two values. The distance from the unstable point and the output of the function is widening as k increases.

For k =3.5,
Orb(f, x, 0.5, 2000, 2030)
= [0.875, 0.3828, 0.8261, 0.5009, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259, 0.5010, 0.8749, 0.3827, 0.8259]

It appears that there are 2 or more unstable steady states for this function that the values are oscillating between.

3.)

<u>For 2 steady-state points:</u>
k = 361
f := MakeNicePol([1, 2, 3, 4, 5, 6, 7, 8], 361, x);

f := 1/500*x^9 - 9/125*x^8 + 273/250*x^7 - 1134/125*x^6 + 22449/500*x^5 - 16821/125*x^4 + 29531/125*x^3 - 27396/125*x^2 + 2041/25*x

SSS(f, x);
[3, 5]

For SSS point x = 3 -
Orb(f, x, 2.6, 1000, 1010);
[2.9999979, 2.9999976, 3.0000021, 3.0000024, 2.9999979, 2.9999976, 3.0000021, 3.0000024, 2.9999979, 2.9999976, 3.0000021]

For SSS point x = 5 -
Orb(f, x, 5.7, 1000, 1010);
[4.9999865, 4.9999959, 5.0000313, 5.0000553, 4.9999547, 5.0000247, 5.0000125, 5.0000145, 5.0000048, 5.0000119, 5.0000305]

<u>For 4 stable steady-state points:</u>
k = 2521
f := MakeNicePol([1, 2, 3, 4, 5, 6, 7, 8], 2521, x);
f := 1/2521*x^9 - 36/2521*x^8 + 546/2521*x^7 - 4536/2521*x^6 + 22449/2521*x^5 - 67284/2521*x^4 + 118124/2521*x^3 - 109584/2521*x^2 + 42841/2521*x

SSS(f, x);
[1, 3, 5, 7]

For SSS point x = 1 -
Orb(f, x, 0.7, 2000, 2010);
[1.00182287, 0.99818918, 1.00181982, 0.99819214, 1.00181684, 0.99819513, 1.00181382, 0.99819808, 1.00181084, 0.99820101, 1.00180787]

For SSS point x = 3 -
Orb(f, x, 2.5, 2000, 2010);
[2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926, 2.99999926]

For SSS point x = 5 -
Orb(f, x, 4.4, 2000, 2010);
[5.00001697, 5.00002394, 5.00000439, 4.99999516, 4.99999631, 5.00000585, 5.00000097, 5.00000004, 4.99999824, 4.99999465, 4.99998964]

For SSS point x = 7 -
Orb(f, x, 7.4, 2000, 2010);
[6.9905529, 7.0092564, 6.9905882, 7.0092033, 6.9905529, 7.0092564, 6.9905882, 7.0092033, 6.9905529, 7.0092564, 6.9905882]