MATH336 - HW #9

1.

characteristic eqn.: $z^2 - 5z + 6 = 0$; $(z-2)(z-3) = 0$; $z_1 = 2, z_2 = 3$

gen. sol.: $C_1 2^n + C_2 3^n$

$$\begin{cases} a(n) = 1 = C_1 2^n + C_2 3^n \\ a(L) = 2 = C_1 2^L + C_2 3^L \end{cases}$$

$$C_1 = \frac{2 - 3^{L-n}}{(2^L - 2^n)(3^{L-n})}$$

$$C_2 = \frac{1 - \left[\frac{2 - 3^{L-n}}{(2^L - 2^n)(3^{L-n})}\right] \cdot 2^n}{3^n}$$

If $L = 1$, $\underbrace{a_n = 2^n}_{a(0) = 1 \text{ and } a(1) = 2}$ ✓ $\boxed{L = 1}$

2.

(i) $\mathbb{P}(\text{success}) = n/L = \frac{700}{1000} = 0.7$; $\boxed{\mathbb{P}(\text{success}) = 0.7}$

(ii) Expected rounds $= n(L-n) = 700(1000-700) = 210{,}000$

(iii) $\mathbb{P}(\text{success}) = \dfrac{1 - (\frac{q}{p})^n}{1 - (\frac{q}{p})^L} = \dfrac{1 - (\frac{0.51}{0.49})^{700}}{1 - (\frac{0.51}{0.49})^{1000}}$

$$\boxed{\mathbb{P}(\text{success}) = 6.134386289 \times 10^{-6}}$$

(iiii) $\mathbb{P}(\text{success}) = \dfrac{1 - (\frac{0.501}{0.499})^{700}}{1 - (\frac{0.501}{0.499})^{1000}}$

$$\boxed{\mathbb{P}(\text{success}) = 0.2881559230}$$

3.

$\mathbb{P}(0)=0, \mathbb{P}(L)=1$

$P(n) = p \cdot \underbrace{P(n+1)}_{\text{win}} + q \cdot \underbrace{P(n-1)}_{\text{lose}}$

$p \cdot P(n+1) + q \cdot P(n-1) - P(n) = 0$

$p \cdot P(n+1) + (1-p)P(n-1) - P(n) = 0$

If $P(n) = z^n$, then $pz^2 - \frac{z}{p} + (1-p) = 0$ and $z = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm \sqrt{(1-2p)^2}}{2p}$

$z = \frac{1 \pm (1-2p)}{2p}$; $z_1 = \frac{2-2p}{2p} = \frac{1-p}{p} = \frac{q}{p}$

$z_2 = \frac{1-(1-2p)}{2p} = \frac{2p}{2p} = 1$; $P(n) = C_1\left(\frac{q}{p}\right)^n + C_2(1)^n = C_1\left(\frac{q}{p}\right)^n + C_2$

$P(0) = 0 = C_1\left(\frac{q}{p}\right)^0 + C_2 = C_1 + C_2$; $C_2 = -C_1$

$P(L) = 1 = C_1\left(\frac{q}{p}\right)^L + C_2$; $C_1\left(\frac{q}{p}\right)^L - C_1 = C_1\left(\left(\frac{q}{p}\right)^L - 1\right)$; $C_1 = \frac{1}{\left(\frac{q}{p}\right)^L - 1}$

and $C_2 = \frac{-1}{\left(\frac{q}{p}\right)^L - 1}$

gen. sol.:

$P(n) = \frac{1}{\left(\frac{q}{p}\right)^L - 1}\left(\frac{q}{p}\right)^n - \frac{1}{\left(\frac{q}{p}\right)^L - 1}$

$= \frac{\left(\frac{q}{p}\right)^n - 1}{\left(\frac{q}{p}\right)^L - 1}$ case when $p \neq q$

for when $p = q = \frac{1}{2}$

characteristic eqn.: $\frac{1}{2}z^2 - z + \frac{1}{2} = 0$

$z^2 - 2z + 1 = 0$; $(z-1)^2$; $z = 1$ (multiplicity = 2)

gen. sol.: $P(n) = C_1 n + C_2$

$P(0) = 0 = C_2$; $C_2 = 0$

$P(L) = C_1(L) + 0 = 1$; $C_1 = \frac{1}{L}$

$P(n) = \frac{1}{L} \cdot n = \frac{n}{L}$

If $p = q = \frac{1}{2}$, $P(n) = \frac{n}{L}$	If $p \neq q$ $P(n) = \frac{\left(\frac{q}{p}\right)^n - 1}{\left(\frac{q}{p}\right)^L - 1}$

4.

$\mathcal{E}$ = expected rounds

$\mathcal{E}(n) = \underbrace{\frac{1}{2}(1+\mathcal{E}_{n+1})}_{\text{win}} + \underbrace{\frac{1}{2}(1+\mathcal{E}_{n-1})}_{\text{lose}} = 1 + \frac{1}{2}\mathcal{E}_{n+1} + \frac{1}{2}\mathcal{E}_{n-1}$, $\mathcal{E}(0)=0$, $\mathcal{E}(L)=0$

$2\mathcal{E}(n) = 2 + \mathcal{E}_{n+1} + \mathcal{E}_{n-1}$; $\mathcal{E}_{n+1} - 2\mathcal{E}_n + \mathcal{E}_{n-1} = -2$

gen. sol. (to homogeneous version): $z^2 - 2z + 1$; $z = 1$ (multiplicity $= 2$)

$\mathcal{E}(n) = C_1(1)^n + C_2(1)^n n = C_1 + C_2 n$

particular sol.: assume $\mathcal{E}(n) = cn^2$

$c(n+1)^2 - 2cn^2 + c(n-1)^2 = -2$

$c(n^2+2n+1) - 2cn^2 + c(n^2+2n+1) = -2$

$\cancel{cn^2} + \cancel{2cn} + c - \cancel{2cn^2} + \cancel{cn^2} - \cancel{2cn} + c = -2$

$2c = -2$; $c = -1$

$\mathcal{E}(n) = -n^2$

full sol.: $\mathcal{E}(n) = C_1 + C_2 n - n^2$

$\mathcal{E}(0) = 0 = C_1$; $C_1 = 0$

$\mathcal{E}(L) = 0 = 0 + C_2 L - L^2 \Rightarrow C_2 L = L^2$; $C_2 = L$

$\mathcal{E}(n) = (0) + Ln - n^2 = n(L-n)$

$\boxed{\mathcal{E}(n) = n(L-n)}$