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Homework for Lecture 7 of Dr. Z.'s Dynamical Models in Biology class

yd372.

Version of Sept. 26 (thanks to Anna Janik, who won a dollar) Email the answers (as a .pdf file) to

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by 8:00pm Monday, Sept. 29, 2025.

Subject: hw7

with an attachment hw7FirstLast.pdf

1. For each of the following non-linear (quadratic) first-order-linear recurrences

(i) Determine all steady-states

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) = \frac{5}{2}\,x(n)\,(1-x(n))$.

b. $x(n+1) = \frac{29}{10}\,x(n)\,(1-x(n))$.

c. $x(n+1) = \frac{31}{10}\,x(n)\,(1-x(n))$.

2.: For each of the following non-linear (cubic) first-order-linear recurrences

(i) Verify that the given points are steady-states

(ii) decide which ones ones are stable

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) = \frac{1}{5}\,x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$

Set of steady-states : $\{1, 2, 3\}$

b. $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

Set of steady-states : $\{2, 3, 5\}$

1. **a.** $x(n+1) = \frac{5}{2}x(n)(1-x(n))$.

(i) Determine all steady-states

no-linear so, set $x = \frac{5}{2}x(1-x)$

$\frac{5}{2}x^2 - \frac{5}{2}x + x = 0$

$\frac{5}{2}x^2 - \frac{3}{2}x = 0$

$x(\frac{5}{2}x - \frac{3}{2}) = 0$

$x = 0$ or $x = \frac{3}{5}$

thus, steady-states : 0 , $\frac{3}{5}$

(ii) decide which of them are stable

Let $f(x) = \frac{5}{2}x(1-x) = \frac{5}{2}x - \frac{5}{2}x^2$

$f'(x) = \frac{5}{2} - 5x$

At $x=0$, $f'(0) = \frac{5}{2} > 1$ unstable

At $x = \frac{3}{5}$, $f'(\frac{3}{5}) = \frac{5}{2} - 5 \times \frac{3}{5} = -\frac{1}{2}$ so. $|f'(3/5)| = 0.5 < 1$, stable

iii) Pick $x_0 = 0.61$ (near 0.6)

$x_1 = 2.5(0.61) \cdot (1-0.61) = 0.59475$

$x_2 = 2.5 \cdot (0.59475)(1-0.59475) \approx 0.602$

so, it converges to 0.6.

Pick $x_0 = 0.01$ (near 0)

$x_1 = 2.5(0.01)(1-0.01) \approx 0.02475$

$x_2 = 2.5(0.02475)(1-0.02475) \approx 0.06038$

moves away from 0.

b. $x(n+1) = \frac{29}{10}x(n)(1-x(n))$

i) $k = 2.9$

set $x = 2.9x(1-x)$

$\Rightarrow 2.9x^2 - 1.9x = 0$

$x(2.9x - 1.9) = 0.$

$x = 0$ or $x = 19/29.$

ii) $f(x) = 2.9x(1-x)$

$f'(x) = 2.9 - 5.8x$

At $x=0$, $f'(0) = 2.9 - 0 = 2.9 > 1$ unstable.

At $x = 19/29$, $f'(19/29) = 2.9 - \frac{58}{10} \times \frac{19}{29} = 2.9 - 3.8 = -0.9$

since $|-0.9| < 1$, so, stable.

iii) Take $x_0 = 0.66$ (near 0.65517)

$x_1 = 2.9(0.66)(1-0.66) = 0.65076$

$x_2 = 2.9(0.65076)(1-0.65076) = 0.65909$

... converges to 0.65517 ($19/29$)

c. $x(n+1) = \frac{31}{10}x(n)(1-x(n))$

$k = \frac{31}{10} = 3.1$

i) set $x = 3.1x(1-x)$

$\Rightarrow 3.1x^2 - 2.1x = 0$

$x(3.1x - 2.1) = 0.$

$x = 0$ or $x = \frac{21}{31}$

ii) Let $f(x) = 3.1x(1-x)$

$f'(x) = 3.1 - 6.2x$

At $x=0$, $f'(0) = 3.1 > 1$ unstable.

At $x = \frac{21}{31}$, $f'(21/31) = 3.1 - 4.2 = -1.1$, $|-1.1| > 1$ unstable.

iii) doesn't converges to $21/31$.

2 **a.** $x(n+1) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$ $\{1, 2, 3\}$

check $x=1$: $0.2 - 1.2 + 3.2 - 1.2 = 1$ yes steady

$x=2$: $1.6 - 4.8 + 6.4 - 1.2 = 2$ yes

$x=3$: $0.2(27) - 1.2(9) + 3.2(3) - 1.2 = 3$ yes

Let $f(x) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$

$f'(x) = \frac{3}{5}x^2 - \frac{12}{5}x + \frac{16}{5}$

At $x=1$, $f'(1) = \frac{3}{5} - \frac{12}{5} + \frac{16}{5} = \frac{7}{5} > 1$ unstable.

At $x=2$, $f'(2) = \frac{12}{5} - \frac{24}{5} + \frac{16}{5} = \frac{4}{5} < 1$ stable.

At $x=3$, $f'(3) = \frac{27}{5} - \frac{36}{5} + \frac{16}{5} = \frac{7}{5} > 1$ unstable

test near 2, converges to 2.

test near 1 or 3, moves away.

b. $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$ $\{2, 3, 5\}$

check $x=2$, $1 - 5 + \frac{39}{4} - \frac{15}{4} = 2$ yes

$x=3$ $\frac{27}{8} - \frac{45}{4} + \frac{3\times39}{8} - \frac{15}{4} = 3$ yes

$x=5$ $\frac{125}{8} - \frac{125}{4} + \frac{39\times5}{8} - \frac{15}{4} = 5$ yes

Let $f(x) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

$f'(x) = \frac{3}{8}x^2 - \frac{5}{2}x + \frac{39}{8}$

At $x=2$, $f'(2) = 1.5 - 5 + 4.875 = 1.375 > 1$ unstable.

At $x=3$ $f'(3) = \frac{27}{8} - \frac{15}{2} + \frac{39}{8} = 0.75 < 1$ stable

At $x=5$ $f'(5) = \frac{75}{8} - \frac{25}{2} + \frac{39}{8} = 1.75 > 1$ unstable

test near 3, converges to 3, otherwise, goes away.