Dynamical Models in Biology — Homework 7

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1. Quadratic (logistic) recurrences

For $x_{n+1} = r\,x_n(1 - x_n)$ with $r > 0$, steady states satisfy

$$x^* = 0 \quad \text{or} \quad x^* = 1 - \frac{1}{r},$$

and stability is by $|f'(x^*)| < 1$ for $f(x) = rx(1-x)$, where $f'(x) = r(1-2x)$.

(a) $r = \frac{5}{2} = 2.5$. Steady states: 0 (unstable since $|f'(0)| = 2.5 > 1$) and 0.6 (stable since $|2 - r| = 0.5 < 1$).

Empirical test (10 steps):

```
r := 5/2:

Logistic := proc(r, x0, N)
    local k, x;
    x := x0;
    seq( evalf(x), k = 0..N ),
    x := r*x*(1-x)
end:

# Near 0 and near 0.6:
Logistic(r, 0.01, 9);
Logistic(r, 0.61, 9);
```

(b) $r = \frac{29}{10} = 2.9$. Steady states: 0 (unstable) and $\frac{19}{29} \approx 0.65517$ (stable since $|2 - r| = 0.9 < 1$).

Empirical test:

```
r := 29/10:
Logistic(r, 0.01, 9);
Logistic(r, 0.66517, 9);
```

(c) $r = \frac{31}{10} = 3.1$. Steady states: 0 (unstable) and $\frac{21}{31} \approx 0.67742$ (unstable since $|2 - r| = 1.1 > 1$).

Empirical test:

```
r := 31/10:
Logistic(r, 0.01, 9);
Logistic(r, 0.679, 9);
```

2. Cubic first-order recurrences

A steady state x^* is stable if $|f'(x^*)| < 1$.

(a) $x_{n+1} = f(x_n)$ with

$$f(x) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}, \qquad f'(x) = \frac{3x^2 - 12x + 16}{5}.$$

Given steady states: $\{1, 2, 3\}$.

$|f'(1)| = \frac{7}{5} > 1 \Rightarrow$ unstable, $\quad |f'(2)| = \frac{4}{5} < 1 \Rightarrow$ stable, $\quad |f'(3)| = \frac{7}{5} > 1 \Rightarrow$ unstable.

Empirical test (10 steps):

```
fa := x -> (x^3 - 6*x^2 + 16*x - 6)/5:

Orbit := proc(f, x0, N)
    local k, x;
    x := x0;
    seq( evalf(x), k = 0..N ),
    x := f(x)
end:

# Near 1, 2, 3:
Orbit(fa, 1.01, 9);
Orbit(fa, 2.01, 9);
Orbit(fa, 3.01, 9);
```

(b) $x_{n+1} = g(x_n)$ with

$$g(x) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}, \qquad g'(x) = \frac{3x^2 - 20x + 39}{8}.$$

Given steady states: $\{2, 3, 5\}$.

$|g'(2)| = \frac{11}{8} > 1 \Rightarrow$ unstable, $\quad |g'(3)| = \frac{6}{8} < 1 \Rightarrow$ stable, $\quad |g'(5)| = \frac{14}{8} > 1 \Rightarrow$ unstable.

Empirical test (10 steps):

```
gb := x -> x^3/8 - (5/4)*x^2 + (39/8)*x - 15/4:

Orbit := proc(f, x0, N)
    local k, x;
    x := x0;
    seq( evalf(x), k = 0..N ),
    x := f(x)
end:

# Near 2, 3, 5:
Orbit(gb, 2.01, 9);
Orbit(gb, 3.01, 9);
Orbit(gb, 5.01, 9);
```