1. For each of the following non-linear (quadratic) first-order-linear recurrences

(i) Determine all steady-states

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) = \frac{5}{2}\, x(n)\,(1 - x(n))$

b. $x(n+1) = \frac{29}{10}\, x(n)\,(1 - x(n))$

c. $x(n+1) = \frac{31}{10}\, x(n)\,(1 - x(n))$

a.

i) $X(n+1) = \frac{5}{2} X(n)(1 - X(n)) \longrightarrow f(X) = \frac{5}{2} X(1-X)$

$$X = \frac{5}{2} X(1-X)$$
$$X - \frac{5}{2} X(1-X) = 0$$
$$X(1 - \frac{5}{2}(1-X)) = 0$$
$$X(1 - \frac{5}{2} + \frac{5}{2}X) = 0$$
$$X(-\frac{3}{2} + \frac{5}{2}X) = 0$$
$$\left.\begin{array}{l} X_1 = 0 \\ X_2 = \frac{3}{5} \end{array}\right\} \text{steady states}$$

ii) $f(X) = \frac{5}{2}X - \frac{5}{2}X^2$

$f'(X) = \frac{5}{2} - 5X$

$f'(0) = \frac{5}{2} - 5(0)$

$= 5/2 > 1 \Rightarrow$ unstable, $X = 0$

$f'(3/6) = \frac{5}{2} - 5(\frac{3}{6})$

$= \frac{5}{2} - 3$

$= -1/2 < 1 \Rightarrow$ stable, $X = \frac{3}{5}$

iii) $X_2 = \frac{3}{5} \Rightarrow X_2' = \frac{4}{5}$; $f(X) = \frac{5}{2} X(1-X)$

$$\left.\begin{array}{l}
f(0) = \frac{4}{5} \\
f(1) = \frac{5}{2}[f(0)](1-[f(0)]) = \frac{5}{2}(\frac{4}{5})(1-\frac{4}{5}) = \frac{2}{5} \\
f(2) = \frac{5}{2}[f(1)](1-[f(1)]) = \frac{5}{2}(\frac{2}{5})(1-\frac{2}{5}) = \frac{3}{5} \\
f(3) = \frac{5}{2}[f(2)](1-[f(2)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(4) = \frac{5}{2}[f(3)](1-[f(3)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(5) = \frac{5}{2}[f(4)](1-[f(4)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(6) = \frac{5}{2}[f(5)](1-[f(5)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(7) = \frac{5}{2}[f(6)](1-[f(6)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(8) = \frac{5}{2}[f(7)](1-[f(7)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(9) = \frac{5}{2}[f(8)](1-[f(8)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5} \\
f(10) = \frac{5}{2}[f(9)](1-[f(9)]) = \frac{5}{2}(\frac{3}{5})(1-\frac{3}{5}) = \frac{3}{5}
\end{array}\right\} \text{converges to } \tfrac{3}{5}\text{; proves s.s.}$$

> $Orb\left(\frac{5}{2} \cdot x \cdot (1-x), x, \frac{3.1}{5}, 0, 10\right)$; $\rightarrow$ converges to $\frac{3}{5}$

[0.6200000000, 0.5890000000, 0.6051975000, 0.5973337150, 0.6013153698, 0.5993379895, 0.6003299095, 0.5998347732, 0.6000825452, 0.5999587102, 0.6000206408] (7)

> $Orb\left(\frac{5}{2} \cdot x \cdot (1-x), x, 0.1, 0, 10\right)$; $\rightarrow$ diverges

[0.1, 0.2250000000, 0.4359375000, 0.6147399902, 0.5920868365, 0.6038000362, 0.5980638812, 0.6009586880, 0.5995183582, 0.6002402410, 0.5998797352] (8)

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b.

i) $X(n+1) = \frac{29}{10} X(n)(1 - X(n)) \longrightarrow f(X) = \frac{29}{10} X(1-X)$

$$X + \frac{29}{10} X(1-X) = 0$$
$$X(1 + \frac{29}{10}(1-X)) = 0$$
$$X(1 + \frac{29}{10} - \frac{29}{10}X) = 0$$
$$X(\frac{19}{10} - \frac{29}{10}X) = 0$$
$$\left.\begin{array}{l} X_1 = 0 \\ X_2 = \frac{19}{29} \end{array}\right\} \text{steady states}$$

ii) $f(X) = \frac{29}{10}X - \frac{29}{10}X^2$

$f'(X) = \frac{29}{10} - \frac{58}{10}X$

$f'(0) = \frac{29}{10} > 1 \Rightarrow$ unstable, $X = 0$

$f'(\frac{19}{29}) = \frac{29}{10} - \frac{58}{10}(-\frac{19}{29})$

$f'(19/29) = \frac{9}{10} < 1 \Rightarrow$ stable, $X = -19/29$

iii)

> $Orb\left(\frac{29}{10} \cdot x \cdot (1-x), x, \frac{19.1}{29}, 0, 10\right)$; $\rightarrow$ converging to 19/29

[0.6586206897, 0.6520344827, 0.6579679967, 0.6526337248, 0.6574385437, 0.6531180045, 0.6570091424, 0.6535095747, 0.6566609504, 0.6538263051, 0.6563786569] (9)

> $Orb\left(\frac{29}{10} \cdot x \cdot (1-x), x, 0.1, 0, 10\right)$; $\rightarrow$ diverging

[0.1, 0.2610000000, 0.5593491000, 0.7147852845, 0.5912151166, 0.7008714272, 0.6079869421, 0.6911825789, 0.6190027424, 0.6839312072, 0.6268910019] (10)

C

i) $X(n+1) = \frac{31}{10} X(n)(1-X(n)) \longrightarrow f(x) = \frac{31}{10} x(1-x)$

$x + \frac{31}{10} x(1-x) = 0$

$x(1+ \frac{31}{10}(1-x) = 0$

$x(1 + \frac{31}{10} - \frac{31}{10} x) = 0$

$\left.\begin{array}{l} x_1 = 0 \\ x_2 = \frac{21}{31} \end{array}\right\}$ S.S.

ii) $f(x) = \frac{31}{10} x - \frac{31}{10} x^2$

$f'(x) = \frac{31}{10} - \frac{62}{10} x$

$f'(0) = \frac{31}{10} > 1 \Rightarrow$ unstable, $x=0$

$f'(21/31) = \frac{31}{10} - \frac{62}{10}\left(\frac{21}{31}\right)$

$f'(21/31) = -11/10 < 1 \Rightarrow$ stable, $x = \frac{21}{31}$

> $Orb\left(\frac{31}{10} \cdot x \cdot (1-x), x, 0.1, 0, 10\right)$; → diverging

[0.1, 0.2790000000, 0.6235929000, 0.7276468648, 0.6143484052, 0.7344657708, 0.6045799871, 0.7410953815, 0.5948063527, 0.7471364420, 0.5856630949] **(11)**

> $Orb\left(\frac{31}{10} \cdot x \cdot (1-x), x, \frac{21.1}{31}, 0, 10\right)$; → converging to 21/31

[0.6806451613, 0.6738387097, 0.6813183193, 0.6730833680, 0.6821306579, 0.6721681128, 0.6831102368, 0.6710589877, 0.6842903504, 0.6697149068, 0.6857102362] **(12)**

2.: For each of the following non-linear (cubic) first-order-linear recurrences

(i) Verify that the given points are steady-states

(ii) decide which ones ones are stable

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{x}$

Set of steady-states : $\{1, 2, 3\}$ ⟶ 1: diverges, 2: converges to 2, 3: diverges

b. $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

Set of steady-states : $\{2, 3, 5\}$ ⟶ 2: diverges, 3: converges to 3, 5: diverges

A)

> $Orb\left(\frac{1}{5} \cdot x^3 - \frac{6}{5} \cdot x^2 + \frac{16}{5} \cdot x - \frac{6}{5}, x, 1.1, 0, 10\right)$

[1.1, 1.134200000, 1.177557595, 1.230784174, 1.293599417, 1.364380503, 1.440144804, 1.517019883, 1.591082973, 1.659191120, 1.719435858] **(13)**

> $Orb\left(\frac{1}{5} \cdot x^3 - \frac{6}{5} \cdot x^2 + \frac{16}{5} \cdot x - \frac{6}{5}, x, 2.1, 0, 10\right)$

[2.1, 2.080200000, 2.064263170, 2.051463614, 2.041198151, 2.032972506, 2.026385174, 2.021111813, 2.016891333, 2.013514031, 2.010811718] **(14)**

> $Orb\left(\frac{1}{5} \cdot x^3 - \frac{6}{5} \cdot x^2 + \frac{16}{5} \cdot x - \frac{6}{5}, x, 3.1, 0, 10\right)$

[3.1, 3.146200000, 3.218129650, 3.336005586, 3.545734670, 3.975231162, 5.121472870, 10.58005045, 135.1920112, 472676.7887, $2.112113789 \times 10^{16}$] **(15)**

B)

> $Orb\left(\frac{1}{8} \cdot x^3 - \frac{5}{4} \cdot x^2 + \frac{39}{8} \cdot x - \frac{15}{4}, x, 2.1, 0, 10\right)$

[2.1, 2.132625000, 2.173856283, 2.224596258, 2.285014295, 2.354172160, 2.429821087, 2.508556886, 2.586391682, 2.659565189, 2.725255038] **(17)**

>

> $Orb\left(\frac{1}{8} \cdot x^3 - \frac{5}{4} \cdot x^2 + \frac{39}{8} \cdot x - \frac{15}{4}, x, 3.1, 0, 10\right)$

[3.1, 3.073875000, 3.054774446, 3.040726348, 3.030345882, 3.022647786, 3.016923178, 3.012657185, 3.009473122, 3.007093730, 3.005314045] **(18)**

> $Orb\left(\frac{1}{8} \cdot x^3 - \frac{5}{4} \cdot x^2 + \frac{39}{8} \cdot x - \frac{15}{4}, x, 5.1, 0, 10\right)$

[5.1, 5.181375000, 5.338712630, 5.669308400, 6.488752370, 9.403012200, 35.49172768, 4183.144593, $9.128095365 \times 10^{9}$, $9.507153776 \times 10^{28}$, $1.074141680 \times 10^{86}$] **(19)**