Homework for Lecture 7 of Dr. Z.'s Dynamical Models in Biology class

Version of Sept. 26 (thanks to Anna Janik, who won a dollar) Email the answers (as a .pdf file) to

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by 8:00pm Monday, Sept. 29, 2025.

Subject: hw7

with an attachment hw7FirstLast.pdf

1. For each of the following non-linear (quadratic) first-order-linear recurrences

(i) Determine all steady-states

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) \,=\, \frac{5}{2}\, x(n)\,(1-x(n))$.

b. $x(n+1) \,=\, \frac{29}{10}\, x(n)\,(1-x(n))$.

c. $x(n+1) \,=\, \frac{31}{10}\, x(n)\,(1-x(n))$.

2.: For each of the following non-linear (cubic) first-order-linear recurrences

(i) Verify that the given points are steady-states

(ii) decide which ones ones are stable

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) \,=\, \frac{1}{5}\, x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$

Set of steady-states : $\{1, 2, 3\}$

b. $x(n+1) \,=\, \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

Note about results:
Maple was oscillating the orbit between 2 values a lot, and I don't know if this is a bug.
code: Orb$\left(\frac{a}{b} * x * (1-x), x, 0.c, 0, 9\right)$;

Set of steady-states : $\{2, 3, 5\}$

(1) $x(n+1) = \frac{5}{2} x(n) (1 - x(n))$

$x = \frac{5}{2} x(1-x) \Rightarrow x = \frac{5}{2}x - \frac{5}{2}x^2 \Rightarrow \frac{3}{2}x - \frac{5}{2}x^2 = 0 \Rightarrow x = 0 \text{ or } \frac{3}{5}$

$f'(x) = \frac{5}{2} - 5x$ for 0: $\frac{5}{2} > 1$, unstable : orbit at 0.1 [0.1, 0.23, 0.44, 0.61, 0.59, 0.60, 0.60, 0.60, 0.60, 0.60] definitely unstable

for $3/5$: $-\frac{1}{2} > -1$, stable : orbit at 0.61 [0.61, 0.595, 0.602, 0.598, 0.601, 0.599, 0.600, 0.599, 0.600, 0.5999] definitely stable

$x(n+1) = \frac{29}{10} x(n) (1 - x(n))$

$x = \frac{29}{10} x(1-x) \Rightarrow \frac{19}{10}x - \frac{29}{10}x^2 = 0 \Rightarrow x = 0 \text{ or } \frac{19}{29}$

$f'(x) = \frac{19}{10} - \frac{58}{10}x$ for 0: $\frac{19}{10} > 1$, unstable: orbit at 0.1 [0.1, 0.26, 0.55, 0.71, 0.59, 0.70, 0.60, 0.69, 0.61] grows away

for $\frac{19}{29}$: using a calculator, $-\frac{9}{10} > -1$, stable: orbit at $\frac{20}{29}$ [0.689, 0.621, 0.682, 0.628, 0.676, 0.634, 0.673, 0.638, 0.670, 0.642] stabilizes in

$x(n+1) = \frac{31}{10} x(n) (1 - x(n))$

$x = \frac{31}{10} x(1-x) \Rightarrow \frac{21}{10}x - \frac{31}{10}x^2 = 0 \Rightarrow x = 0 \text{ or } \frac{21}{31}$

$f'(x) = \frac{21}{10} - \frac{62}{10}x$ for 0: $\frac{21}{10} > 1$, unstable orbit at 0.1 [0.1, 0.28, 0.623, 0.73, 0.61, 0.73, 0.60, 0.74, 0.59, 0.74] definitely unstable

for $\frac{21}{31}$: using a calculator, $-\frac{21}{10} > 1$, unstable. orbit at $\frac{22}{31}$ [0.709, 0.639, 0.715, 0.632, 0.72, 0.624, 0.727, 0.614, 0.734, 0.604] seems to have stability

(2) a) $x(n+1) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$

1: $\frac{1}{5} - \frac{6}{5} + \frac{16}{5} - \frac{6}{5} = \frac{17}{5} - \frac{12}{5} = 1$ ✓

2: $\frac{8}{5} - \frac{24}{5} + \frac{32}{5} - \frac{6}{5} = \frac{40}{5} - \frac{30}{5} = \frac{10}{5} = 2$ ✓ all SS

3: $\frac{27}{5} - \frac{54}{5} + \frac{48}{5} - \frac{6}{5} = \frac{75}{5} - \frac{60}{5} = \frac{15}{5} = 3$ ✓

$f'(x) = \frac{3}{5}x^2 - \frac{12}{5}x + \frac{16}{5}$

1: $\frac{3}{5} - \frac{12}{5} + \frac{16}{5} = \frac{7}{5} > 1$, 1 is unstable

2: $\frac{12}{5} - \frac{24}{5} + \frac{16}{5} = \frac{4}{5} < 1$, 2 is stable

3: $\frac{27}{5} - \frac{36}{5} + \frac{16}{5} = \frac{7}{5}$, 3 is unstable

orbits at x.1:

[1.1, 0.814, 1.31, 0.385, 1.83, -0.801, 1.13, 0.76, 1.39, 0.216] seems chaotic / unstable

[2.1, -1.43, -1.08, 0.333, 1.874, -0.898, 0.886, 1.19, 0.62, 1.57] seems unstable, weirdly

[3.1, -3.57, -22.45, -2867, -4.7×10^{9}, -2.1×10^{28}, -1.88×10^{84}, -1.336×10^{252}, -4.77×10^{755}, -2.17×10^{2266}] definitely runs away

b) $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

$2: \frac{4}{4} - \frac{20}{4} + \frac{39}{4} - \frac{15}{4} = \frac{43}{4} - \frac{35}{4} = \frac{8}{4} = 2 \checkmark$

$3: \frac{27}{8} - \frac{90}{8} + \frac{117}{8} - \frac{30}{8} = \frac{144}{8} - \frac{120}{8} = \frac{24}{8} = 3 \checkmark$ all SS

$5: \frac{125}{8} - \frac{250}{8} + \frac{195}{8} - \frac{30}{8} = \frac{320}{8} - \frac{280}{8} = \frac{40}{8} = 5 \checkmark$

$f'(x) = \frac{3}{8}x^2 - \frac{5}{2}x + \frac{39}{8}$

$2: \frac{3}{2} - 5 + \frac{39}{8} = \frac{11}{8} > 1$, 2 is unstable

$3: \frac{27}{8} - \frac{15}{2} + \frac{39}{8} = \frac{6}{8} < 1$, 3 is stable

$5: \frac{75}{8} - \frac{25}{2} + \frac{39}{8} = \frac{39}{8} > 1$, 5 is unstable

orbits at x.1:

[2.1, 2.13, 2.17, 2.22, 2.28, 2.35, 2.42, 2.50, 2.58, 2.65] definitely unstable

[3.1, 3.07, 3.05, 3.04, 3.03, 3.02, 3.01, 3.01, 3.01, 3.01] definitely stable

[5.1, 5.18, 5.34, 5.67, 6.49, 9.40, 35.49, 4183, 9.12×10^9, 9.51×10^{28}] definitely unstable