Homework for Lecture 7 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Sept. 29, 2025.

Subject: hw7

with an attachment hw7FirstLast.pdf

1. For each of the following non-linear (quadratic) first-order-linear recurrences

(i) Determine all steady-states

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) \,=\, \frac{5}{2}\, x(n)\,(1-x(n)) \quad .$

b. $x(n+1) \,=\, \frac{29}{10}\, x(n)\,(1-x(n)) \quad .$

c. $x(n+1) \,=\, \frac{31}{10}\, x(n)\,(1-x(n)) \quad .$

2.: For each of the following non-linear (cubic) first-order-linear recurrences

(i) Verify that the given points are steady-states

(ii) decide which ones ones are stable

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) \,=\, \frac{1}{5}\, x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{x}$

Set of steady-states : $\{1, 2, 3\}$

b. $x(n+1) \,=\, \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

Set of steady-states : $\{2, 3, 5\}$

1. For each of the following non-linear (quadratic) first-order-linear recurrences

Please don't post!

(i) Determine all steady-states

(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) \;=\; \frac{5}{2}\,x(n)\,(1-x(n))$.

b. $x(n+1) \;=\; \frac{29}{10}\,x(n)\,(1-x(n))$.

c. $x(n+1) \;=\; \frac{31}{10}\,x(n)\,(1-x(n))$.

a) $X(n+1) = \frac{5}{2}X(n)\,(1-X(n))$

i) ↳ underlying function: $f(X) = \frac{5}{2}X(1-X)$
solve $X = f(X)$

$$\cancel{X} = \frac{5}{2}\cancel{X}(1-X)$$
$$1 = \frac{5}{2} - \frac{5}{2}X$$
$$1 - \frac{5}{2} = -\frac{5}{2}X$$
$$-\frac{3}{2} = -\frac{5}{2}X$$
$$\frac{3}{5} = X$$

$\boxed{\text{the steady-states are } X=0 \text{ and } X=\frac{3}{5}}$

ii) $f'(X) = \frac{5}{2} - 5X$

$|f'(0)| = \left|\frac{5}{2}\right| \Rightarrow \frac{5}{2} > 1.$ $X=0$ is <u>unstable</u>

$\left|f'\left(\frac{3}{5}\right)\right| = \left|\frac{5}{2} - 5\left(\frac{3}{5}\right)\right| = \left|-\frac{1}{2}\right| \Rightarrow \frac{1}{2} < 1.$ $X = \frac{3}{5}$ is <u>stable</u>

iii)

```
> f := x -> 5/2 · x · (1 − x)
```

$$f := x \mapsto \frac{5\,x\,(1-x)}{2}$$

```
> orbit := proc(x0, n)
  local i, L;
  L := [x0];
  for i from 1 to n do
  L := [op(L), f(L[−1])];
  end do;
  return L;
  end proc:
orbit(0.01, 10);
```

X(0) X(1) ... X(10)

[0.01, 0.02475000000, 0.06034359375, 0.1417556111, 0.3041523945, 0.5291092885, 0.6228816232, 0.5872502668, 0.6059684772, 0.5969267045, 0.6015130350]

```
orbit(0.7, 10);
```

X(0) X(1) ... X(10)

[0.7, 0.5250000000, 0.6234375000, 0.5869079590, 0.6061175168, 0.5968476815, 0.6015513165, 0.5992183252, 0.6003893098, 0.5998049662, 0.6000974218]

for orbit $(0.01, 10) \Rightarrow X=0$ is the SS, length 10 shows it is diverging away from 0. So, it verifies it is unstable.

for orbit $(0.7, 10) \Rightarrow X = \frac{3}{5}$ is the SS, length 10 shows it is converging back to $0.6 = \frac{3}{5}$. So, it verifies that it is stable.

b) i) $X(n+1) = \frac{29}{10}X(n)(1-X(n))$

↳ underlying function: $f(X) = \frac{29}{10}X(1-X)$
solve $X = f(X)$

$$\cancel{X} = \frac{29}{10}\cancel{X}(1-X)$$
$$1 = \frac{29}{10} - \frac{29}{10}X$$
$$1 - \frac{29}{10} = -\frac{29}{10}X$$
$$\rightarrow -\frac{19}{10} = -\frac{29}{10}X$$
$$\frac{19}{29} = X$$

$\boxed{\text{The SS are } X=0 \text{ and } X=\frac{19}{29}}$

ii) $f'(x) = \frac{29}{10} - \frac{29}{5}x$

$|f'(0)| = \left|\frac{29}{10}\right| > 1 \Rightarrow$ <u>unstable</u>

$\left|f'\left(\frac{19}{29}\right)\right| = \left|\frac{29}{10} - \frac{29}{5}\left(\frac{19}{29}\right)\right| = \left|\frac{29}{10} - \frac{19}{5}\right| = \left|\frac{29}{10} - \frac{38}{10}\right| = \left|\frac{-9}{10}\right| \Rightarrow \frac{9}{10} < 1 \Rightarrow$ <u>stable</u>

iii)

```
> f := x -> 29/10 * x * (1 - x)
```

$f := x \mapsto \frac{29 \cdot x \cdot (1-x)}{10}$

```
> orbit := proc(x0, n)
  local i, L;
  L := [x0];
  for i from 1 to n do
  L := [op(L), f(L[-1])];
  end do;
  return L;
  end proc:
orbit(0.01, 10);
```

[0.01, 0.02871000000, 0.08086863411, 0.2155538046, 0.4903620495, 0.7247306187, 0.5785388321, 0.7071117905, 0.6006036480, 0.6956488274, 0.6139924553]

```
orbit(0.66, 10);
```

[0.66, 0.6507600000, 0.6590871250, 0.6516047314, 0.6583464157, 0.6522865965, 0.6577454982, 0.6528374376, 0.6572580813, 0.6532826981, 0.6568628020]

for orbit $(0.01, 10) \Rightarrow x=0$ is the SS, length 10 shows it is diverging away from 0. So, it verifies it is unstable.

for orbit $(0.66, 10) \Rightarrow x = \frac{19}{29}$ is the SS, length 10 shows it is converging back to $0.65 = \frac{19}{29}$. So, it verifies that it is stable.

c) i) $x(n+1) = \frac{31}{10}x(n)(1-x(n))$

↳ underlying function: $f(x) = \frac{31}{10}x(1-x)$

solve $x = f(x)$

$\cancel{x} = \frac{31}{10}\cancel{x}(1-x)$

$1 = \frac{31}{10} - \frac{31}{10}x$

$1 - \frac{31}{10} = -\frac{31}{10}x$

$-\frac{21}{10} = -\frac{31}{10}x$

$\frac{21}{31} = x$

$\boxed{\text{SS are } x=0 \text{ and } x=\frac{21}{31}}$

ii) $f'(x) = \frac{31}{10} - \frac{31}{5}x$

$|f'(0)| = \frac{31}{10} > 1 \Rightarrow$ <u>unstable</u>

$\left|f'\left(\frac{21}{31}\right)\right| = \left|\frac{31}{10} - \frac{\cancel{31}}{5}\left(\frac{21}{\cancel{31}}\right)\right| = \left|\frac{31}{10} - \frac{42}{10}\right| = \left|\frac{-11}{10}\right| \Rightarrow \frac{11}{10} > 1$ <u>unstable</u>

iii)

```
> f := x -> (31/10)x - 31/10 x^2
```

$f := x \mapsto \frac{31}{10} \cdot x - \frac{31}{10} \cdot x^2$ (1)

```
> orbit := proc(x0, n)
  local i, L;
  L := [x0];
  for i from 1 to n do
  L := [op(L), f(L[-1])];
  end do;
  return L;
  end proc:
orbit(0.01, 10);
```

[0.01, 0.02871000000, 0.08086863411, 0.2155538046, 0.4903620495, 0.7247306187, 0.5785388321, 0.7071117905, 0.6006036480, 0.6956488274, 0.6139924553] (2)

```
orbit(0.7, 10);
```

[0.7, 0.651000000, 0.704316900, 0.645589274, 0.709291666, 0.639210696, 0.714923185, 0.631804876, 0.721145172, 0.623393920, 0.727799216] (3)

for orbit $(0.01, 10) \Rightarrow x=0$ is the SS, length 10 shows it is diverging away from 0. So, it verifies it is unstable.

for orbit $(0.07, 10) \Rightarrow x = \frac{21}{31}$ is the SS, length 10 shows it is diverging away from $0.677 = \frac{21}{31}$. So, it verifies that it is unstable.

2.: For each of the following non-linear (cubic) first-order-linear recurrences

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(ii) decide which of them are stable

(iii) test it empirically, by taking a number close to each steady state, using either a calculator of Maple, find its orbit of length 10 and verify that for stable steady-states, the orbit is attracted to it, but for the unstable ones it runs away.

a. $x(n+1) = \frac{1}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - \frac{6}{5}$

Set of steady-states : $\{1, 2, 3\}$

b. $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$

Set of steady-states : $\{2, 3, 5\}$

2)

a) $X(n+1) = \frac{1}{5}X^3 - \frac{6}{5}X^2 + \frac{16}{5}X - \frac{6}{5}$

$f'(X) = \frac{3}{5}X^2 - \frac{12}{5}X + \frac{16}{5}$

SS: $X=1$

$= \frac{1}{5} - \frac{6}{5} + \frac{16}{5} - \frac{6}{5}$

$= \frac{17}{5} - \frac{12}{5}$

$= \frac{5}{5}$

$= 1$

$X = f(X)$
$1 = f(1)$ ✓

$|f'(1)| = \left|\frac{3}{5} - \frac{12}{5} + \frac{16}{5}\right|$

$= \frac{7}{5} > 1$ unstable

SS: $X=2$

$= \frac{1}{5}(8) - \frac{6}{5}(4) + \frac{16}{5}(2) - \frac{6}{5}$

$= \frac{8}{5} - \frac{24}{5} + \frac{32}{5} - \frac{6}{5}$

$= -\frac{30}{5} + \frac{40}{5}$

$= \frac{10}{5}$

$X = f(X)$
$2 = f(2)$ ✓

$= 2$

$|f'(2)| = \frac{3}{5}(4) - \frac{12}{5}(2) + \frac{16}{5}$

$= \frac{12}{5} - \frac{24}{5} + \frac{16}{5}$

$= \frac{4}{5} < 1$ Stable

SS: $X=3$

$= \frac{1}{5}(27) - \frac{6}{5}(9) + \frac{16}{5}(3) - \frac{6}{5}$

$= \frac{27}{5} - \frac{54}{5} + \frac{48}{5} - \frac{6}{5}$

$= 3$

$X = f(X)$
$3 = f(3)$ ✓

$|f'(3)| = \frac{3}{5}(9) - \frac{12}{5}(3) + \frac{16}{5}$

$= \frac{27}{5} - \frac{36}{5} + \frac{16}{5}$

$= \frac{7}{5} > 1$ unstable

```
> f := x → 1/5 x^3 − 6/5 x^2 + 16/5 x − 6/5
```

$$f := x \mapsto \frac{1}{5} \cdot x^3 - \frac{6}{5} \cdot x^2 + \frac{16}{5} \cdot x - \frac{6}{5}$$

```
> orbit := proc(x0, n)
  local i, L;
  L := [x0];
  for i from 1 to n do
  L := [op(L), f(L[−1])];
  end do;
  return L;
  end proc:
orbit(1.1, 10);
```

[1.1, 1.134200000, 1.177557595, 1.230784174, 1.293599417, 1.364380503, 1.440144804, 1.517019883, 1.591082973, 1.659191120, 1.719435858]

unstable as it diverges away from 1.

```
orbit(2.1, 10);
```

[2.1, 2.080200000, 2.064263170, 2.051463614, 2.041198151, 2.032972506, 2.026385174, 2.021111813, 2.016891333, 2.013514031, 2.010811718]

→ stable converges back to 2.

```
orbit(3.1, 10);
```

[3.1, 3.146200000, 3.218129650, 3.336005586, 3.545734670, 3.975231162, 5.121472870, 10.58005045, 135.1920112, 472676.7887, $2.112113789 \times 10^{16}$]

↳ unstable as it diverges away from 3

b) $x(n+1) = \frac{1}{8}x^3 - \frac{5}{4}x^2 + \frac{39}{8}x - \frac{15}{4}$ $\qquad f'(x) = \frac{3}{8}x^2 - \frac{5}{2}x + \frac{39}{8}$

SS: $x = 2$

$= \frac{1}{8}(8) - \frac{5}{4}(4) + \frac{39}{\cancel{8}_4}(\cancel{2}^1) - \frac{15}{4}$

$= 1 - 5 + \frac{39}{4} - \frac{15}{4}$

$= -4 + \frac{24}{4}$

$= 2$

$x = f(x)$
$2 = f(2)$ ✓

$|f'(2)| = \frac{3}{8}(4) - \frac{5}{2}(2) + \frac{39}{8}$

$= \frac{12}{8} - \frac{40}{8} + \frac{39}{8}$

$= \frac{11}{8} > 1$ unstable

SS: $x = 5$

$= \frac{1}{8}(125) - \frac{5}{4}(25) + \frac{39}{8}(5) - \frac{15}{4}$

$= 5$

$x = f(x)$
$5 = f(5)$ ✓

$|f'(5)| = \frac{3}{8}(25) - \frac{5}{2}(5) + \frac{39}{8}$

$= \frac{7}{4} > 1$ unstable

SS: $x = 3$ $\quad = \frac{1}{8}(27) - \frac{5}{4}(9) + \frac{39}{8}(3) - \frac{15}{4}$

$= 3$

$x = f(x)$
$3 = f(3)$ ✓

$|f'(3)| = \frac{3}{8}(9) - \frac{5}{2}(3) + \frac{39}{8}$

$= \frac{3}{4} < 1$ stable

```
> f := x → 1/8 x^3 − 5/4 x^2 + 39/8 x − 15/4
```

$$f := x \mapsto \frac{1}{8} \cdot x^3 - \frac{5}{4} \cdot x^2 + \frac{39}{8} \cdot x - \frac{15}{4}$$

```
> orbit := proc(x0, n)
  local i, L;
  L := [x0];
  for i from 1 to n do
  L := [op(L), f(L[−1])];
  end do;
  return L;
  end proc:
orbit(2.1, 10);
```

[2.1, 2.132625000, 2.173856283, 2.224596258, 2.285014295, 2.354172160, 2.429821087, 2.508556886, 2.586391682, 2.659565189, 2.725255038]

Unstable as it diverges away from 2

```
orbit(3.1, 10);
```

[3.1, 3.073875000, 3.054774446, 3.040726348, 3.030345882, 3.022647786, 3.016923178, 3.012657185, 3.009473122, 3.007093730, 3.005314045]

→ stable converges to 3.

```
orbit(5.1, 10);
```

[5.1, 5.181375000, 5.338712630, 5.669308400, 6.488752370, 9.403012200, 35.49172768, 4183.144593, $9.128095365 \times 10^{9}$, $9.507153776 \times 10^{28}$, $1.074141680 \times 10^{86}$]

unstable diverges away from 5.