

Dynamical Models in Biology — Homework 6

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1. Random Leslie matrices (5 age classes), classify by eigenvalues.

Let L be a 5×5 Leslie matrix. A population is extinct if $\rho(L) < 1$, critical if $\rho(L) = 1$, and growing if $\rho(L) > 1$, where $\rho(L)$ is the spectral radius.

Maple (auto-generate 10 random Leslie matrices, print and classify):

```
with(LinearAlgebra):

Classify := proc(L)
    local ev, rho;
    ev := Eigenvalues(L);
    rho := max(seq(Re(ev[i]), i=1..nops(ev)));
    if evalf(rho) < 1-10^(-8) then
        "Extinction, rho=" || sprintf("%.6f",evalf(rho))
    elif abs(evalf(rho-1)) <= 10^(-8) then
        "Critical, rho=" || sprintf("%.6f",evalf(rho))
    else
        "Growth, rho=" || sprintf("%.6f",evalf(rho))
    fi;
end proc:

for k to 10 do
    L := RandomLeslieMatrix(5);
    print("---- Matrix " || sprintf("%d",k) || " ----");
    print(Matrix(L));
    print(Classify(L));
od:
```

2. Mini-internet with four sites $S_1, \dots, S_4$.

Staying probabilities: $p_{11} = 0.5$, $p_{22} = 0.4$, $p_{33} = 0.3$, $p_{44} = 0.2$. Each row's remaining probability is split equally among the other three sites.

(a) Transition matrix (rows = from, columns = to):

$$P = \begin{bmatrix} 0.5 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0.2 & 0.4 & 0.2 & 0.2 \\ \frac{7}{30} & \frac{7}{30} & 0.3 & \frac{7}{30} \\ \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & 0.2 \end{bmatrix}.$$

(b) Stationary distribution / PageRank. Solve $\pi P = \pi$, $\sum_i \pi_i = 1$:

$$\pi = \left(\frac{168}{533}, \frac{140}{533}, \frac{120}{533}, \frac{105}{533} \right) \approx (0.3152, 0.2627, 0.2251, 0.1970).$$

Popularity: $S_1 > S_2 > S_3 > S_4$.

Maple (compute π exactly and numerically):

```
with(LinearAlgebra):  
P := Matrix([[1/2,1/6,1/6,1/6],  
             [1/5,2/5,1/5,1/5],  
             [7/30,7/30,3/10,7/30],  
             [4/15,4/15,4/15,1/5]]);  
NS := NullSpace(Transpose(P) - IdentityMatrix(4));  
pi := evalf( NS[1] / add(NS[1][i], i=1..4) ); # normalized stationary  
       $\hookrightarrow$  vector
```