Dynamical Models in Biology — Homework 5

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1. **Recurrence to standard form; companion matrix; compute $a(5)$.**

   (a) Given
$$6\,a(n-1) + a(n+3) + 5\,a(n+1) = 0.$$
   Shift $n \mapsto n+1$ and solve for $a(n+4)$:
$$6\,a(n) + a(n+4) + 5\,a(n+2) = 0 \quad \Longrightarrow \quad \boxed{a(n+4) = -5\,a(n+2) - 6\,a(n)}.$$

   (b) With
$$a(n) = \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix}, \quad a(n+1) = \begin{bmatrix} a(n+4) \\ a(n+3) \\ a(n+2) \\ a(n+1) \end{bmatrix},$$
   and using part (a), we get $a(n+1) = A\,a(n)$ where
$$\boxed{A = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}.$$

   (c) With $a(0) = 0,\ a(1) = 2,\ a(2) = 3,\ a(3) = 4$:
   (i) Standard-form route.
$$a(4) = -5\,a(2) - 6\,a(0) = -5 \cdot 3 - 6 \cdot 0 = -15,$$
$$\boxed{a(5) = -5\,a(3) - 6\,a(1) = -5 \cdot 4 - 6 \cdot 2 = -32}.$$

   (ii) Matrix route. Note that
$$a(0) = \begin{bmatrix} a(3) \\ a(2) \\ a(1) \\ a(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \qquad a(2) = A^2 a(0) = \begin{bmatrix} a(5) \\ a(4) \\ a(3) \\ a(2) \end{bmatrix}.$$
   Compute
$$A^2 = \begin{bmatrix} -5 & 0 & -6 & 0 \\ 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A^2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -32 \\ -15 \\ 4 \\ 3 \end{bmatrix}.$$
   Hence the first component gives $\boxed{a(5) = -32}$, agreeing with part (i).

2. **Leslie matrix and 3-year-olds after two years.**

   Fertility (per female): $f_0 = 0,\ f_1 = 1.5,\ f_2 = 0.9,\ f_3 = 0.5,\ f_4 = 0.3$.
   Survival: $s_0 = 0.8$ (age $0 \to 1$), $s_1 = 0.7$ ($1\to2$), $s_2 = 0.6$ ($2\to3$), $s_3 = 0.6$ ($3\to4$).
   (We take survival from age 4 to the next class as 0.)

   (a) Leslie matrix L (age classes $0, 1, 2, 3, 4$):

$$\boxed{L = \begin{bmatrix} 0 & 1.5 & 0.9 & 0.5 & 0.3 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}.}$$

   (b) Population after two years. Initial vector $x_0 = \begin{bmatrix} 100 \\ 90 \\ 80 \\ 70 \\ 60 \end{bmatrix}$. Then

$$x_1 = Lx_0 = \begin{bmatrix} 260 \\ 80 \\ 63 \\ 48 \\ 42 \end{bmatrix}, \qquad x_2 = Lx_1 = \begin{bmatrix} 213.3 \\ 208 \\ 56 \\ \mathbf{37.8} \\ 28.8 \end{bmatrix}.$$

   Thus the expected number of 3-year-olds after two years is

$$\boxed{37.8 \text{ (individuals, in expectation)} \ \approx 38.}$$