Homework for Lecture 5 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as .pdf file) to

ShaloshBEkhad@gmail.com

by 8:00pm Monday, Sept. 22, 2025.

Subject: hw5

with an attachment hw5FirstLast.pdf

1. a. Convert the recurrence

$$6\,a(n-1) \,+\, a(n+3) \,+\, 5\,a(n+1) = 0 \quad ,$$

into **standard form** where $a(n+4)$ is expressed in terms of $a(n+3), a(n+2), a(n+1), a(n)$.

b.

Abbreviating

$$\mathbf{a}(n) = \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix} \quad .$$

Find the 4×4 matrix, let's call it A such that

$$\mathbf{a}(n+1) \,=\, A\mathbf{a}(n)$$

c Assuming that $a(0) = 0$, $a(1) = 2$, $a(2) = 3$,$a(3) = 4$ Find $a(5)$ in two ways:

(i) Straight from the standard form, by first finding $a(4)$, and then $a(5)$

(ii) Using the matrix version by first finding A^2 and then multiplying it by the column vector $[4, 3, 2, 0]^T$ and extracting the first component.

2. In a certain species only one-year-olds, two-year-olds, three-year-olds, and four-year-olds are fertile. We have

• zero-year-olds can't have babies

• Every 1-year-old female makes 1.5 babies on average

• Every 2-year-old female makes 0.9 babies on average

• Every 3-year-old female makes 0.5 babies on average

- Every 4-year-old female makes 0.3 babies on average

We also know

- The probability that a zero-year-old will survive the year is 0.8
- The probability that a one-year-old will survive the year is 0.7
- The probability that a two-year-old will survive the year is 0.6
- The probability that a three-year-old will survive the year is 0.6

a. Set up the **Leslie matrix**

b. If right now there are 100 zero-year-olds, 90 one-year-olds, 80 two-year-olds, 70 three-year-olds, and 60 four-year-old, what is the expected number of 3-year-olds after two years?

1. a. Convert the recurrence

$$6\,a(n-1) + a(n+3) + 5\,a(n+1) = 0 \quad ,$$

into **standard form** where $a(n+4)$ is expressed in terms of $a(n+3), a(n+2), a(n+1), a(n)$.

$$6a(n-1) + a(n+3) + 5a(n+1) = 0$$

pick largest i, $a(n-i)$ $\quad i = 1$

replace n by $n+1$

$$6a(n) + a(n+4) + 5a(n+2) = 0$$

$$a(n+4) + 0\cdot a(n+3) + 5a(n+2) + 0\cdot a(n+1) + 6a(n) = 0$$

$$\boxed{a(n+4) = 0\cdot a(n+3) - 5a(n+2) + 0\cdot a(n+1) - 6a(n)}$$

$$\begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

b.

Abbreviating

$$\mathbf{a}(n) = \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix} \quad .$$

Find the 4×4 matrix, let's call it A such that

$$\mathbf{a}(n+1) = A\mathbf{a}(n)$$

$$A(n+1) = A \quad a(n)$$

$$a(n+1) = \begin{bmatrix} a(n+4) \\ a(n+3) \\ a(n+2) \\ a(n+1) \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \cdot \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \quad \begin{array}{l} \\ a(n+3) \\ a(n+2) \\ a(n+1) \end{array}$$

c Assuming that $a(0) = 0$, $a(1) = 2$, $a(2) = 3$,$a(3) = 4$ Find $a(5)$ in two ways:

(i) Straight from the standard form, by first finding $a(4)$, and then $a(5)$

(ii) Using the matrix version by first finding A^2 and then multiplying it by the column vector $[4, 3, 2, 0]^T$ and extracting the first component.

(i) $a(n+4) = 0 \cdot a(n+3) - 5a(n+2) + 0 \cdot a(n+1) - 6a(n)$

for $a(4)$, $n=0$

$$a(4) = -5a(2) - 6a(0) = -5(3) - 6(0) = \boxed{-15}$$

for $a(5)$, $n=1$

$$a(5) = -5a(3) - 6a(1) = -5(4) - 6(2) = -20 - 12 = \boxed{-32}$$

(ii) $$A = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \underset{4\times 4}{\begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \cdot \underset{4\times 4}{\begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} = \underset{4\times 4}{\begin{bmatrix} -5 & 0 & -6 & 0 \\ 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

$$= \underset{4\times 4}{\begin{bmatrix} -5 & 0 & -6 & 0 \\ 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \cdot \underset{4\times 1}{\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}} = \underset{4\times 1}{\begin{bmatrix} -32 \\ -15 \\ 4 \\ 3 \end{bmatrix}} \rightarrow 1^{st} \text{ component} = a(5)$$

2. In a certain species only one-year-olds, two-year-olds, three-year-olds, and four-year-olds are fertile. We have

- zero-year-olds can't have babies $f_0 = 0$
- Every 1-year-old female makes 1.5 babies on average $f_1 = 1.5$
- Every 2-year-old female makes 0.9 babies on average $f_2 = .9$
- Every 3-year-old female makes 0.5 babies on average $f_3 = .5$
- Every 4-year-old female makes 0.3 babies on average

We also know

- The probability that a zero-year-old will survive the year is 0.8 $P_0 = .8$
- The probability that a one-year-old will survive the year is 0.7 $P_1 = .7$
- The probability that a two-year-old will survive the year is 0.6 $P_2 = .6$
- The probability that a three-year-old will survive the year is 0.6 $P_3 = .6$

f_i = expected # babies i yr old makes

P_i = probability that i yr old survives to next yr

$$L = \begin{bmatrix} f_0 & f_1 & \cdots & & f_A \\ P_0 & 0 & \cdots & & 0 \\ 0 & P_1 & 0 & \cdots & 0 \\ 0 & 0 & P_2 & \cdots & 0 \\ 0 & \cdots & & P_{A-1} & 0 \end{bmatrix}$$

a. Set up the **Leslie matrix**

$$L = \begin{bmatrix} 0 & 1.5 & .9 & .5 & .3 \\ .8 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & .6 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 \end{bmatrix}$$

b. If right now there are 100 zero-year-olds, 90 one-year-olds, 80 two-year-olds, 70 three-year-olds, and 60 four-year-old, what is the expected number of 3-year-olds after two years?

100 0 yr old
90 1 yr old
80 2 yr old
70 3 yr old
60 4 yr old

\# 3 yr old after 2 yrs = ?

$$\begin{bmatrix} 0 & 1.5 & .9 & .5 & .3 \\ .8 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & .6 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 90 \\ 80 \\ 70 \\ 60 \end{bmatrix} = \begin{array}{c|l} \begin{bmatrix} 213.3 \\ 208 \\ 56 \\ 37.8 \\ 28.8 \end{bmatrix} & \begin{array}{l} 0 \text{ yr old} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$\boxed{\text{37.8 expected 3 yr olds after 2 yrs}}$