1. a. Convert the recurrence

$$6\,a(n-1) \,+\, a(n+3) \,+\, 5\,a(n+1) = 0 \quad ,$$

into **standard form** where $a(n+4)$ is expressed in terms of $a(n+3), a(n+2), a(n+1), a(n)$.

b.

Abbreviating

$$\mathbf{a}(n) = \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix} \quad .$$

Find the 4×4 matrix, let's call it A such that

$$\mathbf{a}(n+1) \,=\, A\mathbf{a}(n)$$

c Assuming that $a(0) = 0$, $a(1) = 2$, $a(2) = 3$, $a(3) = 4$ Find $a(5)$ in two ways:

(i) Straight from the standard form, by first finding $a(4)$, and then $a(5)$

(ii) Using the matrix version by first finding A^2 and then multiplying it by the column vector $[4, 3, 2, 0]^T$ and extracting the first component.

a) $\underbrace{6a(n-1) + a(n+3) + 5a(n+1) = 0}_{\text{pick largest } i=1,\ a(n-1) \to \text{replace } n \text{ w/ } n+1}$

$6a(n) + a(n+4) + 5a(n+2) = 0$

$a(n+4) + 0\cdot a(n+3) + 5a(n+2) + 0\cdot a(n+1) + 6a(n) = 0$

$a(n+4) = 0\cdot a(n+3) - 5a(n+2) + 0\cdot a(n+1) - 6a(n)$

b) $a(n) = \begin{bmatrix} a(n+3) \\ a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix} \longrightarrow \boxed{A = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$

$a(n+3)$
$a(n+2)$
$a(n+1)$

c) $a(0) = 0,\ a(1) = 2,\ a(2) = 3,\ a(3) = 4$

i. $a(n+4) = 0\cdot a(n+3) - 5a(n+2) + 0\cdot a(n+1) - 6a(n)$

<u>for $a(4)$, $n=0$</u> : $a(4) = 0 - 5a(2) + 0 - 6a(0)$

$a(4) = -5(3) - 6(0)$

$a(4) = -15$

<u>for $a(5)$, $n=1$</u> : $a(5) = 0 - 5a(3) + 0 - 6a(1)$

$a(5) = -5(4) - 6(2)$

$a(5) = -32$

ii. $A = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -6 & 0 \\ 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -5 & 0 & -6 & 0 \\ 0 & -5 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -32 \\ -15 \\ 4 \\ 2 \end{bmatrix} \longrightarrow a(5)$

2. In a certain species only one-year-olds, two-year-olds, three-year-olds, and four-year-olds are fertile. We have

f

- zero-year-olds can't have babies — $f_0 = 0$
- Every 1-year-old female makes 1.5 babies on average — $f_1 = 1.5$
- Every 2-year-old female makes 0.9 babies on average — $f_2 = 0.9$
- Every 3-year-old female makes 0.5 babies on average — $f_3 = 0.5$
- Every 4-year-old female makes 0.3 babies on average — $f_4 = 0.3$

We also know

p

- The probability that a zero-year-old will survive the year is 0.8 — $P_0 = 0.8$
- The probability that a one-year-old will survive the year is 0.7 — $P_1 = 0.7$
- The probability that a two-year-old will survive the year is 0.6 — $P_2 = 0.6$
- The probability that a three-year-old will survive the year is 0.6 — $P_3 = 0.6$

a. Set up the **Leslie matrix**

b. If right now there are 100 zero-year-olds, 90 one-year-olds, 80 two-year-olds, 70 three-year-olds, and 60 four-year-old, what is the expected number of 3-year-olds after two years?

a) $$L = \begin{bmatrix} f_0 & f_1 & \cdots & f_A \\ P_0 & 0 & \cdots & 0 \\ 0 & P_1 & \cdots & 0 \\ 0 & 0 & P_{A-1} & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 0 & 1.5 & 0.9 & 0.5 & 0.3 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}$$

b) $$\begin{array}{cl} \begin{bmatrix} 100 \\ 90 \\ 80 \\ 70 \\ 60 \end{bmatrix} & \begin{array}{l} \rightarrow \text{zero} \\ \rightarrow \text{one year old} \\ \rightarrow 2 \\ \rightarrow 3 \\ \rightarrow 4 \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 1.5 & 0.9 & 0.5 & 0.3 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 90 \\ 80 \\ 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 213.3 \\ 208 \\ 56 \\ 37.8 \\ 28.8 \end{bmatrix} \longrightarrow \text{three year olds after 2 years}$$

How many 3 year olds after 2 years?