Dynamical Models in Biology — Homework 4

Praneeth Vedantham (Pv226)

1. **Example of a 2nd-order linear ODE with two solutions; verify linearity.**

   Consider
   $$y'' - y = 0.$$
   Two specific solutions are $y_1(t) = e^t,\ y_2(t) = e^{-t}$. Indeed $y_1'' - y_1 = e^t - e^t = 0, \quad y_2'' - y_2 = e^{-t} - e^{-t} = 0$. By linearity, for any constants c_1, c_2,
   $$y(t) = c_1 e^t + c_2 e^{-t}$$
   satisfies $y'' - y = 0$ since $y'' - y = (c_1 e^t + c_2 e^{-t}) - (c_1 e^t + c_2 e^{-t}) = 0$.

2. **Nonlinear ODE check.** Verify $y_1(t) = t^2$ solves
   $$\big(y'(t)\big)^2 - 4y(t) = 0.$$
   Compute $y_1'(t) = 2t$, so
   $$\big(y_1'(t)\big)^2 - 4y_1(t) = 4t^2 - 4t^2 = 0,$$
   hence y_1 is a solution. Now consider $y_2(t) = 2y_1(t) = 2t^2$. Then
   $$y_2'(t) = 4t, \qquad \big(y_2'(t)\big)^2 - 4y_2(t) = 16t^2 - 8t^2 = 8t^2 \neq 0$$
   for $t \neq 0$, so y_2 is *not* a solution. (Reason: the equation is nonlinear in y' and y, so constant multiples need not solve it.)

3. **Example of a 2nd-order linear recurrence with two solutions; verify linearity.**

   Consider
   $$a(n) = 3a(n-1) - 2a(n-2), \qquad n \geq 2.$$
   Two specific solutions are
   $$a^{(1)}(n) = 1^n \equiv 1, \qquad a^{(2)}(n) = 2^n.$$
   Quick check:
   $$\begin{aligned} &\text{For } a^{(1)}:\ 1 = 3 \cdot 1 - 2 \cdot 1, \\ &\text{For } a^{(2)}:\ 2^n = 3 \cdot 2^{n-1} - 2 \cdot 2^{n-2} = \big(\tfrac{3}{2} - \tfrac{1}{2}\big)2^n = 1 \cdot 2^n. \end{aligned}$$
   Since the recurrence is linear with constant coefficients, any linear combination $c_1 a^{(1)}(n) + c_2 a^{(2)}(n)$ also satisfies it (superposition).

4. **Nonlinear recurrence and a sum of solutions.**

   Given
   $$a(n) = \big(a(n-1)\big)^2, \qquad n \ge 0,$$
   check that
   $$a_1(n) = 2^{\,2^n}, \qquad a_2(n) = 3^{\,2^n}$$
   are solutions:
   $$a_1(n-1) = 2^{\,2^{n-1}} \;\Rightarrow\; \big(a_1(n-1)\big)^2 = \left(2^{\,2^{n-1}}\right)^2 = 2^{\,2\cdot 2^{n-1}} = 2^{\,2^n} = a_1(n),$$
   and the same calculation works with base 3 for a_2. Now consider
   $$a_3(n) = a_1(n) + a_2(n) = 2^{\,2^n} + 3^{\,2^n}.$$
   Plugging into the recurrence would require
   $$a_3(n) \stackrel{?}{=} \big(a_3(n-1)\big)^2 = \left(2^{\,2^{n-1}} + 3^{\,2^{n-1}}\right)^2 = 2^{\,2^n} + 3^{\,2^n} + 2\cdot 2^{\,2^{n-1}}\,3^{\,2^{n-1}},$$
   which includes the cross term $2\cdot 2^{\,2^{n-1}} 3^{\,2^{n-1}} \neq 0$. Therefore
   $$a_3(n) \neq \big(a_3(n-1)\big)^2,$$
   so a_3 is *not* a solution. (Again: the recurrence is nonlinear; sums don't preserve solutions.)

5. **Maple commands and outputs.**

   (a) *IVP ODE:* $y''(x) + y(x) = 0,\ y(0) = 1,\ y'(0) = 1.$
   Maple command:
   ```
   dsolve({diff(y(x),x$2)+y(x)=0, y(0)=1, D(y)(0)=1}, y(x));
   ```
   Output (simplified):
   $$\boxed{y(x) = \cos x + \sin x}.$$

   (b) *IVP difference equation:* $a(n) - 3a(n-1) + a(n-2) = n,\ a(0) = 1,\ a(1) = 3.$
   Maple command:
   ```
   rsolve({a(n)-3*a(n-1)+a(n-2)=n, a(0)=1, a(1)=3}, a(n));
   ```
   Output (one clean closed form): Let $r_1 = \frac{3+\sqrt{5}}{2}$, $r_2 = \frac{3-\sqrt{5}}{2}$. Then
   $$\boxed{a(n) = \Big(1 + \frac{2}{\sqrt{5}}\Big) r_1^n + \Big(1 - \frac{2}{\sqrt{5}}\Big) r_2^n - n - 1}.$$

   Checks: $a(0) = 1,\ a(1) = 3$; also $a(n) - 3a(n-1) + a(n-2) = n$ since the homogeneous part solves $r^2 - 3r + 1 = 0$ and the particular is $-n-1$.

(c) *Eigenvalues/eigenvectors of* $\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$.

Maple command:

```
with(LinearAlgebra):
Eigenvectors(Matrix([[3,4],[2,4]]));
```

Output (simplified): Eigenvalues $\lambda_{1,2} = \dfrac{7 \pm \sqrt{33}}{2}$. A convenient choice of eigenvectors:

$$\lambda_1 = \frac{7-\sqrt{33}}{2}: \quad v^{(1)} = \begin{bmatrix} -(1+\sqrt{33}) \\ 4 \end{bmatrix}, \qquad \lambda_2 = \frac{7+\sqrt{33}}{2}: \quad v^{(2)} = \begin{bmatrix} \sqrt{33}-1 \\ 4 \end{bmatrix}.$$

(Any nonzero scalar multiples are fine.)