Homework for Lecture 4 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as .pdf file) to

ShaloshBEkhad@gmail.com

by 8:00pm Monday, Sept. 22, 2025.

Subject: hw4

with an attachment hw4FirstLast.pdf and/or hw4FirstLast.txt

1. Give an example of a second-order linear differential equation, with two specific functions that are solutions and verify that their sum also satisfies that same differential equation.

2. Verify that $y_1(t) = t^2$ satisfies the differential equation

$$y'(t)^2 - 4y(t) = 0 \quad .$$

Would you expect the function $y_2(t) = 2y_1(t) = 2t^2$ to also be a solution? (after all it is a constant multiple of $y_1(t)$). Explain. Verify that indeed $y_2(t)$ is **not** a solution.

3. Give an example of a second-order linear recurrence equation, with two specific sequences that are solutions and verify that their sum also satisfies that same recurrence equation.

4. Consider the non-linear recurrence

$$a(n) = a(n-1)^2 \quad , n \geq 0 \quad .$$

Check that both sequences

$$a_1(n) := 2^{2^n} \quad , \quad a_2(n) := 3^{2^n} \quad ,$$

are solutions. Does it follow that the new sequence

$$a_3(n) := a_1(n) + a_2(n) = 2^{2^n} + 3^{2^n} \quad ,$$

is automatically yet-another-solution? Explain why or why not. By directly plugging-in into the recurrence find out whether it is true.

5. Write the Maple commands to solve each of the following problems, and give the Maple output.

a. Solve the Initial Value Problem Differential Equation

$$y''(x) + y(x) = 0 \quad , \quad y(0) = 1 \quad , y'(0) = 1 \quad .$$

. b Solve the Initial Value Problem Difference Equation

$$a(n) - 3a(n-1) + a(n-2) \; = n \quad a(0) = 1, a(1) = 3 \quad .$$

c. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$$

1. Give an example of a second-order linear differential equation, with two specific functions that are solutions and verify that their sum also satisfies that same differential equation.

- 2^{nd} order linear dif eqy
- 2 functions as solutions
- sum of functions is also a sol

$$r^2 - 3r + 2 = 0$$
$$(r-2)(r-1) = 0$$
$$r = 2, 1$$
$$y = c_1 e^x + c_2 e^{2x}$$

$$\boxed{y'' - 3y' + 2y = 0}$$

$$\boxed{y = c_1 e^x,\ y = c_2 e^{2x},\ \text{and } y = c_1 e^x + c_2 e^{2x} \text{ are all solutions}}$$

<u>Check:</u>

$y_1 = c_1 e^x$
$y_1' = c_1 e^x$
$y_1'' = c_1 e^x$

plug in

$y'' - 3y' + 2y \stackrel{?}{=} 0$
$c_1 e^x - 3c_1 e^x + 2c_1 e^x = 0$ ✓

$y_2 = c_2 e^{2x}$
$y_2' = 2c_2 e^{2x}$
$y_2'' = 4c_2 e^{2x}$

$y'' - 3y' + 2y \stackrel{?}{=} 0$
$4c_2 e^{2x} - 6c_2 e^{2x} + 2c_2 e^{2x} = 0$ ✓

$y_3 = c_1 e^x + c_2 e^{2x}$
$y_3' = c_1 e^x + 2c_2 e^{2x}$
$y_3'' = c_1 e^x + 4c_2 e^{2x}$

<u>plug in</u> $y'' - 3y' + 2y \stackrel{?}{=} 0$

$c_1 e^x + 4c_2 e^{2x} - 3c_1 e^x - 6c_2 e^{2x} + 2c_1 e^x + 2c_2 e^{2x} \stackrel{?}{=} 0$

$0 = 0$ ✓

2. Verify that $y_1(t) = t^2$ satisfies the differential equation

$$y'(t)^2 - 4y(t) = 0 \quad .$$

$y_1(t) = t^2$
$y'(t) = 2t$

$(2t)^2 - 4(t^2) \stackrel{?}{=} 0$
$4t^2 - 4t^2 = 0$ ✓

Would you expect the function $y_2(t) = 2y_1(t) = 2t^2$ to also be a solution? (after all it is a constant multiple of $y_1(t)$). Explain. Verify that indeed $y_2(t)$ is **not** a solution.

$y_2(t)$ isnt necessarily a sol

$y_2'(t) = 4t$

$(4t)^2 - 4(2t^2) \stackrel{?}{=} 0$
$16t^2 - 8t^2 \neq 0$

$y_2(t)$ is <u>not</u> a sol

3. Give an example of a second-order linear recurrence equation, with two specific sequences that are solutions and verify that their sum also satisfies that same recurrence equation.

$a(n) = 3a(n-1) + 4a(n-2)$

$z^2 - 3z - 4 = 0$
$(z-4)(z+1) = 0$
$z = 4, -1$

$a_1(n) = (4)^n + (-1)^n$
$a_2(n) = 4^n$
$a_3(n) = (-1)^n$

need to verify that all are solutions

$a_2(n) = 4^n$
$a(n) \stackrel{?}{=} 3a(n-1) + 4a(n-2)$
$4^n \stackrel{?}{=} 3 \cdot 4^{n-1} + 4 \cdot 4^{n-2}$
$4^n \stackrel{?}{=} 3 \cdot 4^n \cdot 4^{-1} + 4 \cdot 4^n \cdot 4^{-2}$
$4^n \stackrel{?}{=} \frac{3}{4} \cdot 4^n + \frac{4}{16} \cdot 4^n$
$4^n = 4^n$ ✓ $a_2(n)$ is a sol

$a_3(n) = (-1)^n$
$a(n) = 3 \cdot a(n-1) + 4a(n-2)$
$(-1)^n \stackrel{?}{=} 3 \cdot (-1)^{n-1} + 4(-1)^{n-2}$
$(-1)^n \stackrel{?}{=} 3(-1)^n(-1)^{-1} + 4(-1)^n(-1)^{-2}$
$(-1)^n = -3(-1)^n + 4(-1)^n$
$(-1)^n = (-1)^n$ ✓ $a_3(n)$ is a sol

$a_1(n) = 4^n + (-1)^n$
$a(n) = 3a(n-1) + 4a(n-2)$
$4^n + (-1)^n \stackrel{?}{=} 3 \cdot 4^{n-1} + 4 \cdot 4^{n-2} + 3 \cdot (-1)^{n-1} + 4(-1)^{n-2}$
$4^n + (-1)^n \stackrel{?}{=} 3 \cdot 4^n \cdot 4^{-1} + 4 \cdot 4^n \cdot 4^{-2} + 3(-1)^n(-1)^{-1} + 4(-1)^n(-1)^{-2}$
$4^n + (-1)^n \stackrel{?}{=} \frac{3}{4} \cdot 4^n + \frac{4}{16} \cdot 4^n - 3(-1)^n + 4(-1)^n$
$4^n + (-1)^n = 4^n + (-1)^n$

$a_1(n)$, which is the sum of a_2 and a_3 is also a solution.

4. Consider the non-linear recurrence

$$a(n) = a(n-1)^2 \quad , n \geq 0 \quad .$$

Check that both sequences

$$a_1(n) := 2^{2^n} \quad , \quad a_2(n) := 3^{2^n} \quad ,$$

are solutions. Does it follow that the new sequence

$$a_3(n) := a_1(n) + a_2(n) = 2^{2^n} + 3^{2^n} \quad ,$$

is automatically yet-another-solution? Explain why or why not. By directly plugging-in into the recurrence find out whether it is true.

$a_1(n) := 2^{2^n}$

plug into $a(n) = a(n-1)^2$

$2^{2^n} \stackrel{?}{=} \left(2^{2^{(n-1)}}\right)^2$

$2^{2^n} \stackrel{?}{=} 2^{2 \cdot 2^{(n-1)}}$

$2^{2^n} \stackrel{?}{=} 2^{2 \cdot 2^n \cdot 2^{-1}}$ $\quad 2 \cdot 2^{-1} = 1$

$2^{2^n} = 2^{2^n}$ ✓ equivalent

$a_2(n) := 3^{2^n}$

plug into $a(n) = a(n-1)^2$

$3^{2^n} \stackrel{?}{=} \left(3^{2^{(n-1)}}\right)^2$

$3^{2^n} \stackrel{?}{=} 3^{2 \cdot 2^{(n-1)}}$

$3^{2^n} \stackrel{?}{=} 3^{2 \cdot 2^n \cdot 2^{-1}}$

$3^{2^n} = 3^{2^n}$ ✓ equivalent

the sum isnt necessarily still a solution → need to check if still linear recurrence

$a_3(n) := 2^{2^n} + 3^{2^n}$

$a(n) \stackrel{?}{=} a(n-1)^2$

$2^{2^n} + 3^{2^n} \stackrel{?}{=} \left(2^{2^{(n-1)}} + 3^{2^{(n-1)}}\right)^2 \rightarrow$ foil $\left(2^{2^{n-1}}\right)^2 + 2\left(2^{2^{n-1}} \cdot 3^{2^{n-1}}\right) + \left(3^{2^{n-1}}\right)^2$

$\underbrace{2^{2^n} + 3^{2^n}}$ $\stackrel{?}{=}$ $\underbrace{2^{2^n} + 2 \cdot 2^{2^{n-1}} \cdot 3^{2^{n-1}} \quad 3^{2^n}}$

these are NOT equivalent, therefore a_3 isnt another solution

5. Write the Maple commands to solve each of the following problems, and give the Maple output.

a. Solve the Initial Value Problem Differential Equation

$$y''(x) + y(x) = 0 \quad , \quad y(0) = 1 \quad , y'(0) = 1 \quad .$$

```
dsolve({diff(y(x),x,x) + y(x)=0, y(0)=1, D(y)(0)=1}, y(x));
```

$y(x) = \cos(x) + \sin(x)$

. b Solve the Initial Value Problem Difference Equation

$$a(n) - 3a(n-1) + a(n-2) = n \quad a(0) = 1, a(1) = 3 \quad .$$

```
rsolve({a(n) - 3*a(n-1) + a(n-2)-n=0, a(0)=1, a(1)=3}, a(n));
```

```
rsolve( {a(n) - 3·a(n - 1) + a(n - 2) - n = 0, a(0) = 1, a(1) = 3}, a(n));
```

$$\left(\frac{1}{2} - \frac{3\sqrt{5}}{10}\right)\left(-\frac{\sqrt{5}}{2} + \frac{3}{2}\right)^n + \left(\frac{3\sqrt{5}}{10} + \frac{1}{2}\right)\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)^n - \frac{(\sqrt{5}+1)\sqrt{5}\left(-\frac{2}{-3-\sqrt{5}}\right)^n}{5(-3-\sqrt{5})} - \frac{(\sqrt{5}-1)\sqrt{5}\left(-\frac{2}{\sqrt{5}-3}\right)^n}{5(\sqrt{5}-3)} - n - 1 \qquad (2)$$

c. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$$

```
with(linalg);
A := matrix([[3,4],[2,4]]);
eigenvals(A);
```

$\frac{7+\sqrt{33}}{2}, \frac{7-\sqrt{33}}{2}$

```
eigenvects(A);
```

$$\left[\frac{7}{2} + \frac{\sqrt{33}}{2}, 1, \left\{\left[1 \quad \frac{1}{8} + \frac{\sqrt{33}}{8}\right]\right\}\right], \left[\frac{7}{2} - \frac{\sqrt{33}}{2}, 1, \left\{\left[1 \quad \frac{1}{8} - \frac{\sqrt{33}}{8}\right]\right\}\right]$$