1. Give an example of a second-order linear differential equation, with two specific functions that are solutions and verify that their sum also satisfies that same differential equation.

$y''+y=0$; $y_1(x)=\sin x$, $y_2(x)=\cos x$

$y_1''=-\sin x \Rightarrow y_1''+y_1=-\sin x+\sin x=0$
$y_2''=-\cos x \Rightarrow y_2''+y_2=-\cos x+\cos x=0$ $\Big\}$ $y(x)=-\sin x-\cos x$

$y''+y=(-\sin x-\cos x)+(\sin x+\cos x)=0$ ✓

2. Verify that $y_1(t)=t^2$ satisfies the differential equation

$$y'(t)^2-4y(t)=0 \quad .$$

Would you expect the function $y_2(t)=2y_1(t)=2t^2$ to also be a solution? (after all it is a constant multiple of $y_1(t)$). Explain. Verify that indeed $y_2(t)$ is **not** a solution.

$y_1(t)=t^2$
$y_1'(t)=2t$ $\Big\}$ $[y'(t)]^2-4y(t)=0$
$(2t)^2-4(t^2)=0$
$4t^2-4t^2=0$
$0=0$ ✓ $\Big\}$ $y_1(t)$ is a sol.

$y_2(t)=2y_1(t)=2t^2$
$y_2'(t)=4t$ $\Big\}$ $[y'(t)]^2-4y(t)=0$
$(4t)^2-4(2t^2)=0$
$16t^2-8t^2\neq 0$ X $\Big\}$ $y_2(t)$ is not a sol

3. Give an example of a second-order linear recurrence equation, with two specific sequences that are solutions and verify that their sum also satisfies that same recurrence equation.

$a_n=4a_{n-1}-4a_{n-2}$ $(n\geq 2)$
$z^2=4z-4$
$(z-2)^2=0$
$z=2$ $\Big\}$ gen sol: $a_n=C_1(2)^n+C_2n(2)^n$

$u_n=2^n$, $v_n=2^n\cdot n$

$a_n=4u_{n-1}-4u_{n-2}$
$=4(2^{n-1})-4(2^{n-2})$
$=2^{n-2}(4\cdot 2-4)$
$=2^{n-2}\cdot 4$
$=2^n=u_n$ ✓

$a_n=4v_{n-1}-4v_{n-2}$
$=4((n-1)2^{n-1})-4((n-2)2^{n-2})$
$=2^{n-2}(4(n-1)\cdot 2-4(n-2))$
$=2^{n-2}(8n-8-4n+8)$
$=2^{n-2}(4n)$
$=n\cdot 2^n=v_n$ ✓

verify $w_n=u_n+v_n \Rightarrow w_n=2^n+n2^n$
$a_n=4w_{n-1}-4w_{n-2}$
$=(4u_{n-1}-4u_{n-2})+(4v_{n-1}-4v_{n-2})$
$=u_n+v_n=w_n$

gen sol: $a_n=C_1(2)^n+C_2n(2)^n$

4. Consider the non-linear recurrence

$$a(n) = a(n-1)^2 \quad , n \geq 0 \quad .$$

Check that both sequences

$$a_1(n) := 2^{2^n} \quad , \quad a_2(n) := 3^{2^n} \quad ,$$

are solutions. Does it follow that the new sequence

$$a_3(n) := a_1(n) + a_2(n) = 2^{2^n} + 3^{2^n} \quad ,$$

is automatically yet-another-solution? Explain why or why not. By directly plugging-in into the recurrence find out whether it is true.

$a_1(n) := 2^{2^n} \longrightarrow a(n) = a(n-1)^2 = (2^{2^{n-1}})^2 = 2^{2 \cdot 2^{n-1}} = 2^{2^n} = a_1(n)$ ✓

$a_2(n) := 3^{2^n} \longrightarrow a(n) = a(n-1)^2 = (3^{2^{n-1}})^2 = 3^{2 \cdot 2^{n-1}} = 3^{2^n} = a_2(n)$ ✓

$a_3(n-1)^2 = [a_1(n-1) + a_2(n-1)^2]$

$= a_1(n-1)^2 + \underbrace{2a_1(n-1)a_2(n-1)}_{\text{extra term}}$

X $a_1(n) + a_2(n) = a_3(n)$

→ failure b/c recurrance is non-linear

5. Write the Maple commands to solve each of the following problems, and give the Maple output.

a. Solve the Initial Value Problem Differential Equation

$$y''(x) + y(x) = 0 \quad , \quad y(0) = 1 \quad , y'(0) = 1 \quad .$$

. b Solve the Initial Value Problem Difference Equation

$$a(n) - 3a(n-1) + a(n-2) = n \quad a(0) = 1, a(1) = 3 \quad .$$

c. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$$

a) dsolve({diff(y(x), x$2) + y(x) = 0, y(0) = 1, D(y)(0) = 1}, y(x));

output: y(x) = sin(x) + cos(x)

b) rsolve({a(n) - 3·a(n-1) + a(n-2) = n, a(0) = 1, a(1) = 3}, a(n));

output: $\left(\frac{1}{2} - \frac{3\sqrt{5}}{10}\right)\left(-\frac{\sqrt{5}}{2} + \frac{3}{2}\right)^n + \left(\frac{3\sqrt{5}}{10} + \frac{1}{2}\right)\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)^n - \left(\frac{(\sqrt{5}+1)\sqrt{5}\left(\frac{-2}{-3-\sqrt{5}}\right)^n}{5(-3-\sqrt{5})}\right) - \left(\frac{(\sqrt{5}-1)\sqrt{5}\left(-\frac{2}{\sqrt{5}-3}\right)^n}{5(\sqrt{5}-3)}\right) - n - 1$

c) with(LinearAlgebra)
A := matrix([[3,4],[2,4]]);

output: $A := \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$

eigenvals(A);

output: $\left\{\frac{7}{2} + \frac{\sqrt{33}}{2}, \frac{7}{2} - \frac{\sqrt{33}}{2}\right\}$

eigenvects(A);

output: $\left[\frac{7}{2} + \frac{\sqrt{33}}{2}, 1, \left\{\left[1, \frac{1}{8} + \frac{\sqrt{33}}{8}\right]\right\}\right],$
$\left[\frac{7}{2} - \frac{\sqrt{33}}{2}, 1, \left\{\left[1, \frac{1}{8} - \frac{\sqrt{33}}{8}\right]\right\}\right]$