HW 4: Dyn mod in Bio

1. Give an example of a second-order linear differential equation, with two specific functions that are solutions and verify that their sum also satisfies that same differential equation.

$y'' + y = 0$

$\left.\begin{array}{l} y_1(x) = \sin(x) \\ y_2(x) = \cos(x) \end{array}\right\}$ sol'ns

verify each is a sol'n

$y_1(x) = \sin(x)$
$y_1'(x) = \cos(x)$
$y_1''(x) = -\sin(x)$

$y'' + y = 0$
$-\sin(x) + \sin(x) = 0$
$0 = 0$ ✓

$y_2(x) = \cos(x)$
$y_2'(x) = -\sin(x)$
$y_2''(x) = -\cos(x)$

$y'' + y = 0$
$\cos(x) - \cos(x) = 0$
$0 = 0$ ✓

$y_3(x) = \sin(x) + \cos(x)$
$y'(x) = \cos(x) - \sin(x)$
$y''(x) = -\sin(x) - \cos(x)$

$y'' + y = -\sin(x) - \cos(x) + \sin(x) + \cos(x) = 0$
$0 = 0$ ✓

2. Verify that $y_1(t) = t^2$ satisfies the differential equation

$$y'(t)^2 - 4y(t) = 0 \quad .$$

Would you expect the function $y_2(t) = 2y_1(t) = 2t^2$ to also be a solution? (after all it is a constant multiple of $y_1(t)$). Explain. Verify that indeed $y_2(t)$ is **not** a solution.

$y_1(t) = t^2$
$y_1'(t) = 2t$

verify it is a sol'n

$(y'(t))^2 - 4y(t) = 0$
$(2t)^2 - 4(t^2) = 0$
$4t^2 - 4t^2 = 0$
$0 = 0$ ✓

$y_1(t)$ is a sol'n

$y_2(t) = 2y_1(t) = 2t^2$
$y_2'(t) = 4t$

verify: is it a sol'n?

$(y'(t))^2 - 4y(t) = 0$
$(4t)^2 - 4(2t^2) = 0$
$16t^2 - 8t^2 \neq 0$ X

$y_2(t)$ is not a sol'n

3. Give an example of a second-order linear recurrence equation, with two specific sequences that are solutions and verify that their sum also satisfies that same recurrence equation.

$a_n = -a_{n-1} + 2a_{n-2} \quad (n \geq 2)$

$z^2 = -z + 2$

$z^2 + z - 2 = 0$

$(z+2)(z-1) = 0$

$z = \{-2, 1\} \rightarrow$ the roots

gen sol'n: $a_n = C_1(-2)^n + C_2$

call the two sol'ns: $U_n = 1^n = 1$ and $V_n = (-2)^n$

Verify U_n is a sol'n:

$$a_n = -U_{n-1} + 2U_{n-2} = -1 + 2 = 1 = U_n \checkmark$$

Verify V_n is a sol'n:

$$\begin{aligned} a_n = -V_{n-1} + 2V_{n-2} &= -(-2)^{n-1} + 2(-2)^{n-2} \\ &= (-2)^{n-2}\left(-(-2)^1 + 2\right) \\ &= (-2)^{n-2}(2+2) \\ &= 4(-2)^{n-2} = (-2)^n = V_n \end{aligned}$$

$W_n = U_n + V_n = 1 + (2)^n$

using linearity $\rightarrow$
$$\begin{aligned} -W_{n-1} + 2W_{n-2} &= -(U_{n-1} + V_{n-1}) + 2(U_{n-2} + V_{n-2}) \\ &= -U_{n-1} + 2U_{n-2} + V_{n-1} + 2V_{n-2} \\ &= U_n + V_n \\ &= W_n \checkmark \end{aligned}$$

gen sol'n. $a_n = C_1(-2)^n + C_2$

4. Consider the non-linear recurrence

$$a(n) = a(n-1)^2 \quad , n \geq 0 \quad .$$

Check that both sequences

$$a_1(n) := 2^{2^n} \quad , \quad a_2(n) := 3^{2^n} \quad ,$$

are solutions. Does it follow that the new sequence

$$a_3(n) := a_1(n) + a_2(n) = 2^{2^n} + 3^{2^n} \quad ,$$

check $a_1(n) := 2^{2^n}$

$$a(n) = a(n-1)^2 = \left(2^{2^{n-1}}\right)^2 = 2^{2 \cdot 2^{n-1}} = 2^{2^n} = a_1(n) \checkmark$$

check $a_2(n) := 3^{2^n}$

$$a(n) = a(n-1)^2 = \left(3^{2^{n-1}}\right)^2 = 3^{2 \cdot 2^{n-1}} = 3^{2^n} = a_2(n) \checkmark$$

$$a_3(n-1)^2 = \left(a_1(n-1) + a_2(n-1)\right)^2 = a_1(n-1)^2 + 2a_1(n-1)a_2(n-1).$$

$a_1(n) + a_2(n) = a_3(n)$, extra term that isn't generally $= 0$.

for example: $n=0$ $\quad a_1(0) = 2,\ a_2(0) = 3 \rightarrow a_3(0) = 5$

the recurrence predicts $a_3(1) = 5^2 = 25$, but $a_1(1) + a_2(1) = 2^2 + 3^2 = 15 \neq 25$

The failure is b/c the recurrence is non-linear, so superposition doesn't hold.

5. Write the Maple commands to solve each of the following problems, and give the Maple output.

a. Solve the Initial Value Problem Differential Equation

$$y''(x) + y(x) = 0 \quad , \quad y(0) = 1 \quad , y'(0) = 1 \quad .$$

input: dsolve({diff(y(x), x$2) + y(x) = 0, y(0) = 1, D(y)(0) = 1}, y(x));

output: $y(x) = \sin(x) + \cos(x)$

b Solve the Initial Value Problem Difference Equation

$$a(n) - 3a(n-1) + a(n-2) = n \quad a(0) = 1, a(1) = 3 \quad .$$

input: rsolve({a(n) - 3·a(n-1) + a(n-2) - n = 0, a(0) = 1, a(1) = 3}, a(n));

output:

$$\left(\frac{1}{2} - \frac{3\sqrt{5}}{10}\right)\left(-\frac{\sqrt{5}}{2} + \frac{3}{2}\right)^n + \left(\frac{3\sqrt{5}}{10} + \frac{1}{2}\right)\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)^n - \frac{(\sqrt{5}+1)\sqrt{5}\left(-\frac{2}{-3-\sqrt{5}}\right)^n}{5(-3-\sqrt{5})} - \frac{(\sqrt{5}-1)\sqrt{5}\left(-\frac{2}{\sqrt{5}-3}\right)^n}{5(\sqrt{5}-3)} - n - 1$$

c. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$$

with(LinearAlgebra)

input: A := matrix([[3,4], [2,4]]);

output: $A := \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$

Input: eigenvals(A);

output $\left\{\frac{7}{2}+\frac{\sqrt{33}}{2}, \frac{7}{2}-\frac{\sqrt{33}}{2}\right\}$

Input: eigenvects(A);

output: $\left[\frac{7}{2}+\frac{\sqrt{33}}{2}, 1, \left\{\left[1 \; \frac{1}{8}+\frac{\sqrt{33}}{8}\right]\right\}\right], \left[\frac{7}{2}-\frac{\sqrt{33}}{2}, 1, \left\{\left[1 \; \frac{1}{8}-\frac{\sqrt{33}}{8}\right]\right\}\right]$