Dynamical Models in Biology — Homework 3

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1. **(a) Prove that** $a_1(n) = 2^{2^n}$ **satisfies** $a(n) = \big(a(n-1)\big)^2$.
   Proof (induction / direct check). For any $n \geq 1$,

$$a_1(n-1) = 2^{2^{n-1}} \quad \Longrightarrow \quad \big(a_1(n-1)\big)^2 = \Big(2^{2^{n-1}}\Big)^2 = 2^{\,2 \cdot 2^{n-1}} = 2^{\,2^n} = a_1(n).$$

   So $a_1(n)$ satisfies the recurrence.

   Is $a_2(n) = 3 \cdot 2^{2^n}$ **also a solution? Why/why not?**
   Compute the right-hand side using a_2:

$$\big(a_2(n-1)\big)^2 = \big(3 \cdot 2^{2^{n-1}}\big)^2 = 9 \cdot 2^{\,2^n},$$

   but $a_2(n) = 3 \cdot 2^{\,2^n}$. Since $9 \cdot 2^{\,2^n} \neq 3 \cdot 2^{\,2^n}$, it fails the recurrence. *Reason:* the equation is non-linear; constant multiples of a solution are not solutions (unless the constant is 1 or 0).

2. **Solve** $a(n) = 3a(n-1) - 2a(n-2) + n$ with $a(0) = 2,\ a(1) = 3$.

   Homogeneous part: $r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$, so

$$a_h(n) = C_1 \cdot 1^n + C_2 \cdot 2^n = C_1 + C_2\, 2^n.$$

   Particular for the polynomial input n: because $r = 1$ is a root (multiplicity 1), try a quadratic $a_p(n) = An^2 + Bn$. Applying $L[a] = a(n) - 3a(n-1) + 2a(n-2)$ gives

$$L[An^2 + Bn] = (-2A)n + (5A - B).$$

   Match $L[a_p] = n$ to get $-2A = 1 \Rightarrow A = -\frac{1}{2}$ and $5A - B = 0 \Rightarrow B = -\frac{5}{2}$. Thus

$$a(n) = C_1 + C_2\, 2^n - \frac{1}{2}n^2 - \frac{5}{2}n.$$

   Use $a(0) = 2$ and $a(1) = 3$:

$$\begin{cases} C_1 + C_2 = 2, \\ C_1 + 2C_2 - \frac{1}{2} - \frac{5}{2} = 3 \end{cases} \quad \Rightarrow\ C_2 = 4,\ C_1 = -2.$$

   Therefore

$$\boxed{a(n) = 4 \cdot 2^n - \frac{n^2 + 5n + 4}{2}}.$$

   (Checks: $n = 0 \Rightarrow 2$, $n = 1 \Rightarrow 3$.)

3. **Solve** $a(n) = 2a(n-1) + 2a(n-2) - 2a(n-3) + 3$ with $a(0) = 3,\ a(1) = 2,\ a(2) = 6$.

First get a constant particular: try $a_p(n) \equiv C$. Plugging in

$$C = 2C + 2C - 2C + 3 \ \Rightarrow\ -C = 3 \ \Rightarrow\ C = -3.$$

So let $b(n) := a(n) + 3$. Then b satisfies the homogeneous recurrence

$$b(n) - 2b(n-1) - 2b(n-2) + 2b(n-3) = 0,$$

with initial data $b(0) = 6,\ b(1) = 5,\ b(2) = 9$.

Characteristic equation:

$$r^3 - 2r^2 - 2r + 2 = 0.$$

It has three distinct real roots (no simple rationals):

$$r_1 \approx -1.1700864866, \quad r_2 \approx 0.6888921825, \quad r_3 \approx 2.4811943041.$$

Hence

$$b(n) = \alpha\, r_1^n + \beta\, r_2^n + \gamma\, r_3^n, \quad \text{so} \quad a(n) = \alpha\, r_1^n + \beta\, r_2^n + \gamma\, r_3^n - 3.$$

Solving the 3×3 system from $b(0) = 6,\ b(1) = 5,\ b(2) = 9$ yields

$$\alpha \approx 0.5016785140, \quad \beta \approx 4.4944413145, \quad \gamma \approx 1.0038801715.$$

Thus a convenient closed form is

$$\boxed{a(n) \ = \ 0.50168\,(-1.170087)^n \ + \ 4.49444\,(0.688892)^n \ + \ 1.003880\,(2.481195)^n \ - \ 3}.$$

Quick check: This gives $a(0) = 3$, $a(1) = 2$, $a(2) = 6$, and subsequent terms $a(3) = 13$, $a(4) = 37$, $a(5) = 91$, matching the recurrence.