Homework for Lecture 3: Ezra Chechik

1. (a) prove that $a_1(n) = 2^{2^n}$ satisfies the non-linear recurrence equation.

$a_n = a(n-1)^2$

$2^{2^n} = (2^{2^{n-1}})^2 = 2^{2^1 \cdot 2^{n-1}} = 2^{2^n}$ ✓

$3 \cdot 2^{2^n}$ is not a solution to the non-linear recurrence eq. While by inspect this is obvious, one must also note that the superposition principle does not apply to non-linear recurrences.

2. $a_n = 3a_{n-1} - 2a_{n-2} \boxed{+n}$; $a(0)=2$, $a(1)=3$

I. $a_n - 3a_{n-1} + 2a_{n-2} = n$

solve $a_{n\ homo}$ $\rightarrow$ CE $\rightarrow z^2 - 3z + 2 = 0$

$(z-1)(z-2) = 0$

thus, $a_{n\,gen\,homo} = c_1(1)^n + c_2(2)^n$

II. a_n part.? guess form $a_n = \alpha n + \beta$

$\alpha n + \beta - 3(\alpha(n-1) + \beta) + 2(\alpha(n-2) + \beta) \stackrel{?}{=} n$

$\alpha n + \beta \underline{-3\alpha n} + 3\alpha - 3\beta + \underline{2\alpha n} - 4\alpha + 2\beta = n$

$\longrightarrow$ Guess Fails

new guess: $a_{n\,part.} = \alpha n^2 + \beta n + \Delta$

Next page $\longrightarrow$

• $a_n = \alpha n^2 + \beta n + \Delta$

• $-3a_{n-1} = -3[\alpha(n-1)^2 + \beta(n-1) + \Delta]$
$= -3[\alpha(n^2-2n+1) + \beta(n-1) + \Delta]$
$= -3\alpha n^2 + 6\alpha n - 3\alpha - 3\beta n + 3\beta - 3\Delta$

• $2a_{n-2} = 2[\alpha(n-2)^2 + \beta(n-2) + \Delta]$
$= 2[\alpha(n^2-4n+4) + \beta(n-2) + \Delta]$
$= 2\alpha n^2 - 8\alpha n + 8\alpha + 2\beta n - 4\beta + 2\Delta$

Σ terms $\overset{?}{=} n$, for α, β, Δ

If $a_n = \alpha n^2 + \beta n + \Delta$ is a solution:

• $\alpha n^2 - 3\alpha n^2 + 2\alpha n^2 = 0n^2$ ✓
• $\cancel{\beta n} + 6\alpha n - \cancel{3\beta n} - 8\alpha n + \cancel{2\beta n} = n$
$\rightarrow -2\alpha n = n$, $\alpha = -1/2$
• $\cancel{\Delta} - 3\alpha + 3\beta - \cancel{3\Delta} + 8\alpha - 4\beta + \cancel{2\Delta} = 0$
$\rightarrow 5\alpha - \beta = 0$
$5(-1/2) - \beta = 0 \rightarrow -\frac{5}{2} - \beta = 0$, $\beta = -\frac{5}{2}$

$a_{n\,part} = -\frac{1}{2}n^2 - \frac{5}{2}n + \Delta$

$a_{n\,gen} = c_1 + c_2(2)^n - \frac{1}{2}n^2 - \frac{5}{2}n + \Delta$

$2 = c_1 + c_2 + \Delta$
$3 = c_1 + 2c_2 - \frac{1}{2} - \frac{5}{2} + \Delta$
$6 = c_1 + 2c_2 + \Delta$

$$c_1 + c_2 + \Delta = 2$$

$$- \; c_1 + 2c_2 + \Delta = 6$$

$$-c_2 = -4, \; c_2 = 4$$

$$c_1 = 2 - c_2 - \Delta = -2 - \Delta$$

$$a_{n\,\text{part}_{\text{nonhomo}}} = (-2 - \cancel{\Delta}) + 4(2)^n - \frac{1}{2}n^2 - \frac{5}{2}n + \cancel{\Delta}$$

$$\boxed{a_{n\,\text{part}_{\text{nh}}} = 4(2)^n - \frac{1}{2}n^2 - \frac{5}{2}n - 2}$$

3. Solve:

$$a_n = 2a_{n-1} + 2a_{n-2} - 2a_{n-3} + 3$$

IC: $a(0) = 3, \; a(1) = 2, \; a(2) = 6$

I: $a_n - 2a_{n-1} + 2a_{n-2} + 2a_{n-3} = 0$

$$z^3 - 2z^2 + 2z + 2 = 0$$

$$z^2(z-2) - 2(z-2) = 0$$

$$(z-2)(z^2-2) \longrightarrow z = 2, \; z = +\sqrt{2}, \; z = -\sqrt{2}$$

$$a_{n\,\text{gen}_{\text{homo}}} = c_1(2)^n + c_2(\sqrt{2})^n + c_2(-\sqrt{2})^n$$

a_n part => guess $a_n = \alpha$

$$\alpha - \cancel{2\alpha} + \cancel{2\alpha} + 2\alpha = 3$$

$$3\alpha = 3, \; \alpha = 1$$

$$\boxed{a(n)_{\text{gen}_{\text{nh}}} = c_1(2)^n + c_2(\sqrt{2})^n + c_2(-\sqrt{2})^n + 1}$$