Dyn Mod in Bio HW3

1. (a) Prove that $a_1(n) = 2^{2^n}$ satisfies the non-linear recurrence equation

$$a(n) = a(n-1)^2 \quad .$$

Is the following constant multiple of the seqeuncec $a_1(n)$, given by $a_2(n) = 3 \cdot 2^{2^n}$, also a solution? Why?

$2^1 \cdot 2^{n-1} = 2^n$

$$(a_1(n-1))^2 = \left(2^{2^{n-1}}\right)^2 = 2^{2 \cdot 2^{n-1}} = 2^{2^n} \checkmark$$

Thus, $a_1(n)$ satisfies $a(n) = (a(n-1))^2$

check if $a_2(n) = 3 \cdot 2^{2^n}$ is a sol'n. → plug it in

$$(a_2(n-1))^2 = \left(3 \cdot 2^{2^{n-1}}\right)^2 = 9 \cdot 2^{2^n}$$

$9 \neq 3$

more generally $a(n) = c \cdot 2^{2^n}$, c is a constant.
$c^2 = c$ just another constant, so c is forced to be either 1 or 0, to satisfy the reoccurrence. Thus, $a(n) = 3 \cdot 2^{2^n}$ is not a sol'n.

2. Solve the following recurrence with the given initial conditions

degree 1 → P.S. must be degree 2 (add 1 more)

$$a(n) = 3a(n-1) - 2a(n-2) + n \quad ; \quad a(0) = 2 \quad , \quad a(1) = 3 \quad .$$

$$a(n) - 3a(n-1) + 2a(n-2) = n$$

homog part: $a(n) - 3a(n-1) + 2a(n-2) = 0$

$$z^2 - 3z + 2 = 0$$
$$(z-2)(z-1) = 0$$
$$z = \{1, 2\}$$
$$a_h(n) = c_1 1^n + c_2 2^n$$
$$\hookrightarrow a_h(n) = c_1 + c_2 2^n$$

find a P.S. $a(n) = An^2 + Bn$

$$An^2 + Bn - \left[\left(3\left(A(n-1)^2 + B(n-1)\right) - 2\left(A(n-2)^2 + B(n-2)\right)\right)\right] = n$$

$$\cancel{An^2} + \cancel{Bn} - \cancel{3An^2} + 6An - 3A - \cancel{3Bn} + 3B + \cancel{2An^2} + 8An + 8A + \cancel{2Bn} - 4B = n$$

$$-2An + 5A - B = n \qquad 5A - B = 0$$

$$-2An = n \qquad 5\left(-\tfrac{1}{2}\right) - B = 0$$

$$-2A = 1 \qquad B = \frac{-5}{2}$$

$$A = -\frac{1}{2}$$

general sol'n: $a(n) = C_1 + C_2 2^n - \frac{1}{2}n^2 - \frac{5}{2}n$

$$a(0) = 2 = C_1 + C_2 \qquad 2 = C_1 + C_2$$

$$a(1) = 3 = C_1 + 2C_2 - 3 \rightarrow 6 = C_1 + 2C_2$$

solve system of eqn

$$\begin{array}{r} 2 = \cancel{C_1} + C_2 \\ -[6 = \cancel{C_1} + 2C_2] \\ \hline -4 = -C_2 \end{array} \qquad C_2 = 4$$

$$C_1 = -2$$

sol'n: $\boxed{a(n) = -2 + 4 \cdot 2^n - \frac{1}{2}n^2 - \frac{5}{2}n}$

3. Solve the following recurrence with the given initial conditions

$$a(n) = 2a(n-1) + 2a(n-2) - 2a(n-3) + 3 \quad ; \quad a(0) = 3 \quad , \quad a(1) = 2 \quad , \quad a(2) = 6 \quad .$$

$a(n) - 2a(n-1) - 2a(n-2) + 2a(n-3) = 3$

homog part: $z^3 - 2z^2 - 2z + 2 = 0$

$z_1 = -1.17 \,,\, z_2 = 0.689 \,,\, z_3 = 2.48$

homog. gen sol'n: $a(n) = C_1 z_1^n + C_2 z_2^n + C_3 z_3^n$

non-homog. part $a_p(n) = An + B$

$An + B - 2(A(n-1)+B) - 2(A(n-2)+B) + 2(A(n-3)+B) = 3$

$An + B - 2An + \cancel{2A} - 2B \quad \cancel{-2An} + \cancel{4A} \cancel{-2B} + \cancel{2An} - \cancel{6A} + \cancel{2B} = 3$

$-An - B = 3$

$A = 0$

$B = -3$

gen sol'n: $a(n) = C_1 z_1^n + C_2 z_2^n + C_3 z_3^n - 3$

$a(0) = 3 = C_1 + C_2 + C_3 - 3$ $\qquad z_1 = -1.17 \,,\, z_2 = 0.689 \,,\, z_3 = 2.48$

$a(1) = 2 = -1.17C_1 + 0.689C_2 + 2.48C_3 - 3$

$a(2) = 6 = C_1(-1.17)^2 + C_2(0.689)^2 + C_3(2.48)^2 - 3$

$6 = C_1(1.3689) + C_2(0.4747) + C_3(6.1504) - 3$

used online system of eqn $\}$ $C_1 = 0.5021 \,,\, C_2 = 4.493 \,,\, C_3 = 1.00476$

sol'n: $\boxed{a(n) = 0.5021(-1.17)^n + 4.493(0.689)^n + 1.00476(2.48)^n - 3}$