Homework for Lecture 21 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (either as .pdf file and/or .txt file) to

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by 8:00pm Monday, Dec. 1,, 2025.

Subject: hw21

with an attachment hw21FirstLast.pdf and/or hw21FirstLast.txt

1. By hand solve the system

$$\frac{dx}{dt} = x - y \quad , \frac{dy}{dt} = y - x \quad , \quad x(0) = 1 \quad , \quad y(0) = 1 \quad .$$

Plot, by hand, the phase-plane diagram.

2. Now use Maple with the command

```
S:=dsolve({diff(x(t),t)=x(t)-y(t),diff(y(t),t)=y(t)-x(t),x(0)=1,y(0)=0},{x(t),y(t)});
plot([subs(S,x(t)),subs(S,y(t)),t=0..10]);
```

did you get the same thing?

3. Use Maple to solve and then plot the phase-plane diagram for the system

$$\frac{dx}{dt} = a_{11}x + a_{12}y \quad , \frac{dy}{dt} = a_{21}x + a_{22}y \quad , \quad x(0) = 1 \quad , \quad y(0) = 1 \quad ,$$

for three randomly chosen matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad .$$

4. Carefully read, and understand, the Maple code for the following procedures (type Help(ProcedureName); for instructions)

`Lotka, Volterra, VolterraM`

in the Maple package

`https://sites.math.rutgers.edu/~zeilberg/Bio25/DMB.txt` ,

For **each of them**, experiment with **three** randmom choices of parameters, and random initial conditions, using `Dis` (with $h = 0.01$), of *each* of the quantities in question.

Send me these nice plots.

Confirm the numerics by using `SEquP`.

1. By hand solve the system

$$\frac{dx}{dt} = x - y \quad , \frac{dy}{dt} = y - x \quad , \quad x(0) = 1 \quad , \quad y(0) = 1 \quad .$$

Plot, by hand, the phase-plane diagram.

$$x' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$(1-\lambda)(1-\lambda) - 1 = 0$$

$$1 - \lambda - \lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - 2\lambda = 0$$
$$\lambda(\lambda - 2) = 0$$
$$\lambda = 0$$
$$\lambda = 2$$

If $\lambda = 0$ $\begin{bmatrix} 1-0 & -1 \\ -1 & 1-0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\left.\begin{matrix} V_1 - V_2 = 0 \\ V_1 = 1 \\ V_2 = 1 \end{matrix}\right\}$ if $\lambda = 0$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

if $\lambda = 2$ $\begin{bmatrix} 1-2 & -1 \\ -1 & 1-2 \end{bmatrix} \to \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$ $\left.\begin{matrix} -V_1 - V_2 = 0 \\ V_1 = -1 \\ V_2 = 1 \end{matrix}\right\}$ If $\lambda = 2$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

general sol'n: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$

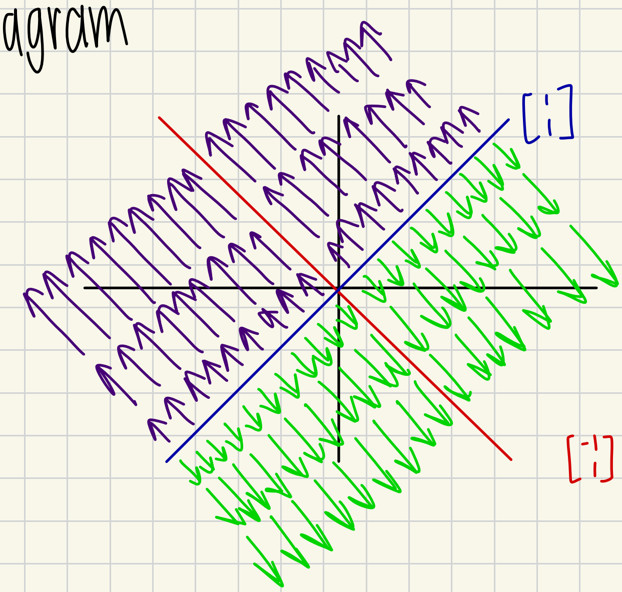
plug in initial cond. $\left.\begin{matrix} x(0) = 1 \\ y(0) = 1 \end{matrix}\right\}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0$

$$1 = c_1 - c_2$$
$$+ (1 = c_1 + c_2)$$
$$2 = 2c_1$$

$c_1 = 1 \quad c_2 = 1$

sol'n: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

phase diagram



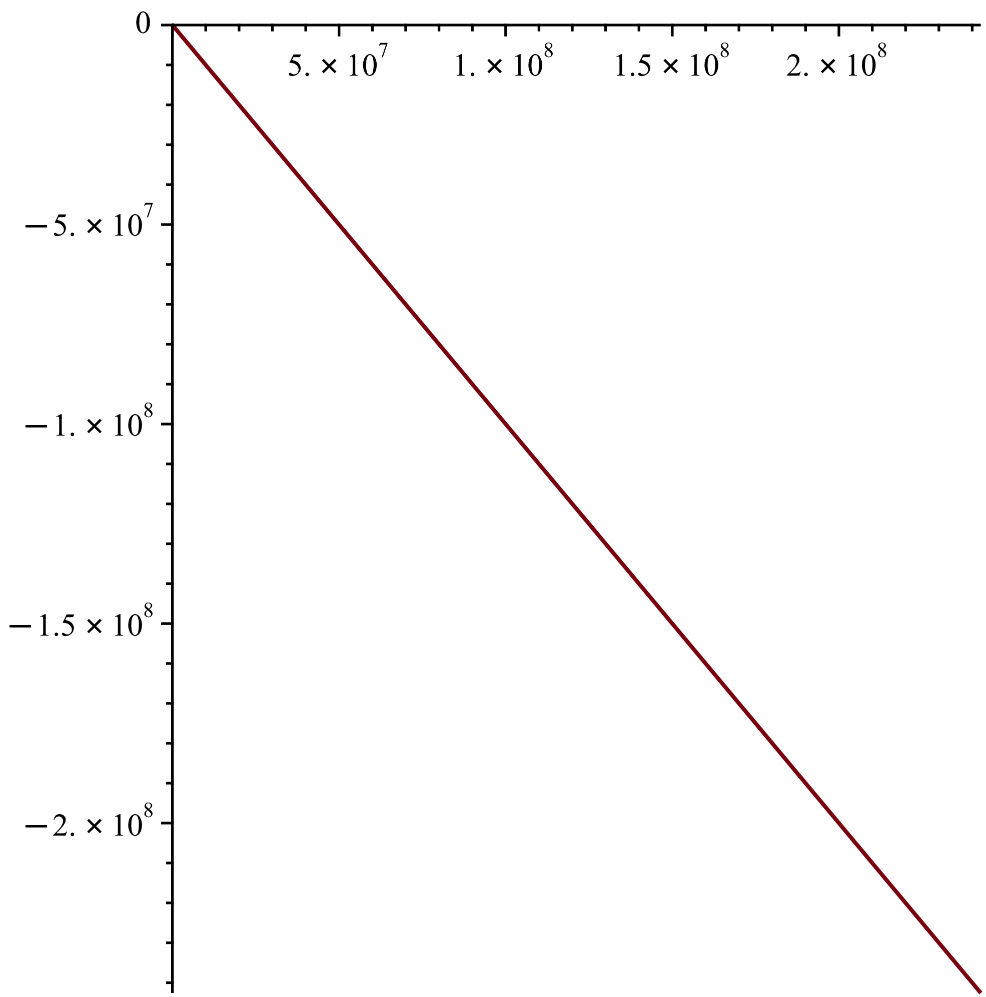
Problem 2

```
> read`DMB.txt`
```

For a list of the Main procedures type: Help(); for help with a specific procedure type: Help(ProcedureName); for example Help(Feig);
For a list of the Continuous Dynamical Models procedures type: HelpC(); for help with a specific procedure type: Help(ProcedureName); for example Help(Feig); **(1)**

```
> S := dsolve( {diff(x(t), t) = x(t) - y(t), diff(y(t), t) = y(t) - x(t), x(0) = 1, y(0) = 0},
      {x(t), y(t)} );
  plot( [subs(S, x(t)), subs(S, y(t)), t = 0 ..10] );
```

$$S := \left\{ x(t) = \frac{1}{2} + \frac{e^{2t}}{2}, y(t) = -\frac{e^{2t}}{2} + \frac{1}{2} \right\}$$




```
> with(linalg)
```

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, **(2)** *eigenvects*, *entermatrix*, *equal*, *exponential*, *extend*, *ffgausselim*, *fibonacci*, *forwardsub*, *frobenius*, *gausselim*, *gaussjord*, *geneqns*, *genmatrix*, *grad*, *hadamard*, *hermite*, *hessian*, *hilbert*, *htranspose*, *ihermite*, *indexfunc*, *innerprod*, *intbasis*, *inverse*, *ismith*, *issimilar*, *iszero*, *jacobian*, *jordan*, *kernel*, *laplacian*, *leastsqrs*, *linsolve*, *matadd*, *matrix*, *minor*, *minpoly*, *mulcol*, *mulrow*, *multiply*, *norm*, *normalize*, *nullspace*, *orthog*, *permanent*, *pivot*, *potential*, *randmatrix*, *randvector*, *rank*, *ratform*, *row*, *rowdim*, *rowspace*, *rowspan*, *rref*, *scalarmul*, *singularvals*, *smith*, *stackmatrix*, *submatrix*, *subvector*, *sumbasis*, *swapcol*, *swaprow*, *sylvester*, *toeplitz*, *trace*, *transpose*, *vandermonde*, *vecpotent*, *vectdim*, *vector*, *wronskian*]

>

Problem 3

```
> restart :

with(plots) :
with(LinearAlgebra) :

print("=== Phase-Plane Plot for 3 Random Matrices ===") :

for k to 3 do

  print(" ") :
  print("----------- Matrix ", k, " -----------") :

  # Random matrix
  A := RandomMatrix(2, 2, generator=-3..3) :
  print(A);

  # ODE system
  sys := {
    diff(x(t), t) = A[1, 1]*x(t) + A[1, 2]*y(t),
    diff(y(t), t) = A[2, 1]*x(t) + A[2, 2]*y(t),
    x(0) = 1, y(0) = 1 } :
  sol := dsolve(sys) :
  print("Solution:", sol);

  # Extract explicit solutions x(t), y(t)
  X := unapply( eval(x(t), sol), t ) :
  Y := unapply( eval(y(t), sol), t ) :

  # Parametric phase-plane plot (this ALWAYS works)
  p := plot( [X(t), Y(t), t=0..10],
        color=blue, thickness=2,
        title=cat("Phase-Plane Trajectory for Matrix ", k) ) :

  display(p);

end do;
```

"=== Phase-Plane Plot for 3 Random Matrices ==="

" "

"----------- Matrix ", 1, " -----------"

$$A := \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$$

$$sys := \left\{ \frac{d}{dt}\, x(t) = 3\,x(t) - 3\,y(t),\ \frac{d}{dt}\, y(t) = 3\,x(t) + 3\,y(t), x(0) = 1, y(0) = 1 \right\}$$

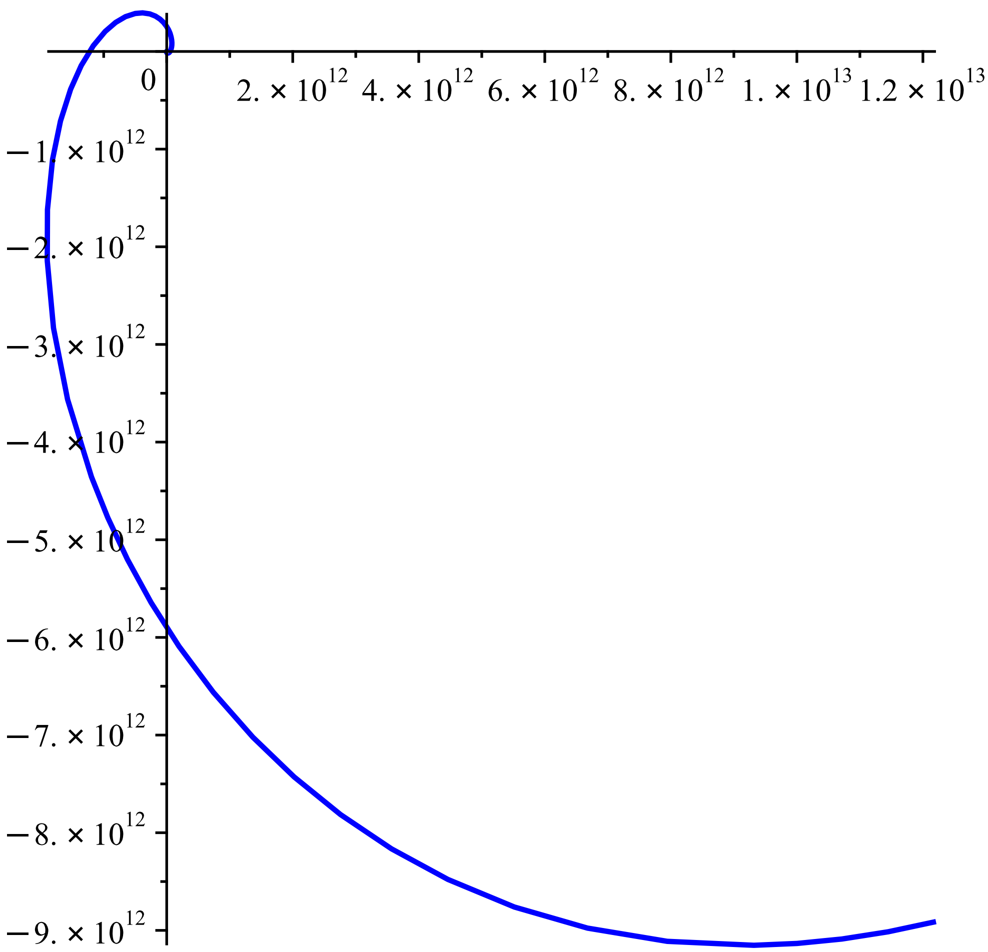
$$sol := \left\{ x(t) = e^{3\,t}\,(\cos(3\,t) - \sin(3\,t)), y(t) = -e^{3\,t}\,(-\cos(3\,t) - \sin(3\,t)) \right\}$$

$$\text{"Solution:"}, \left\{ x(t) = e^{3\,t}\,(\cos(3\,t) - \sin(3\,t)), y(t) = -e^{3\,t}\,(-\cos(3\,t) - \sin(3\,t)) \right\}$$

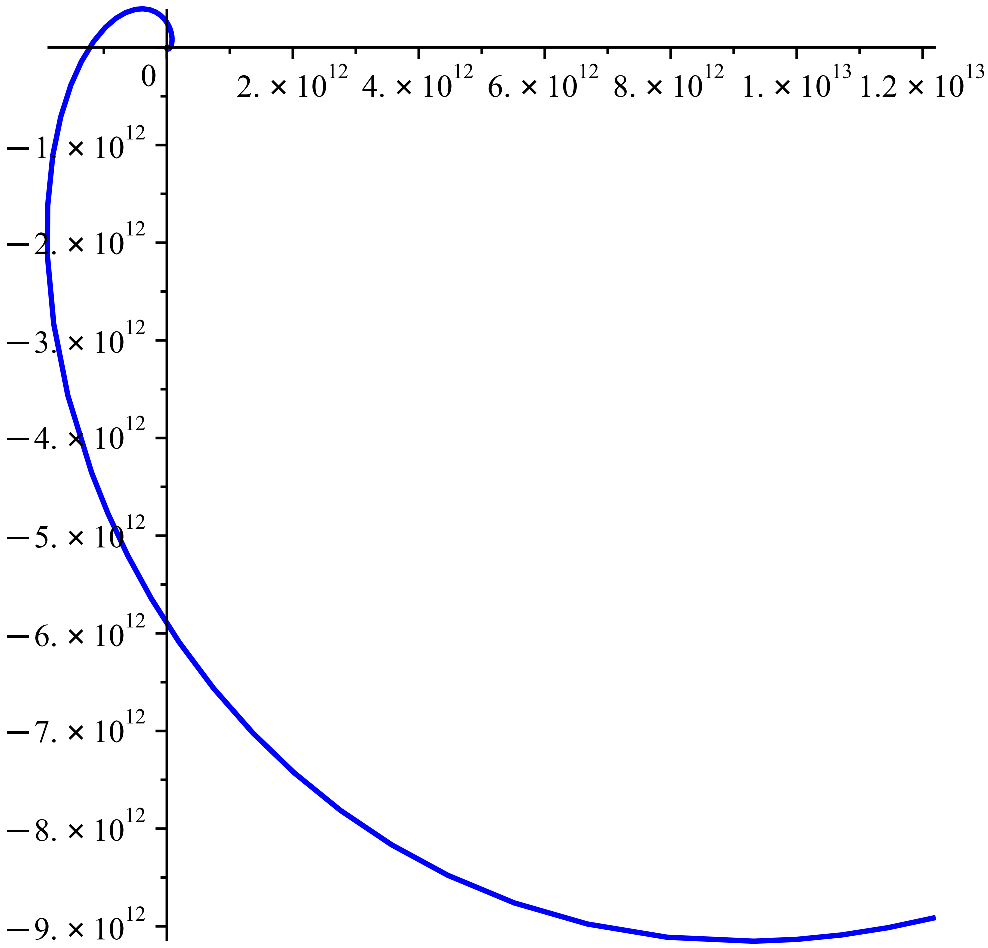
$$X := t \mapsto e^{3\cdot t}\cdot(\cos(3\cdot t) - \sin(3\cdot t))$$

$$Y := t \mapsto -e^{3\cdot t}\cdot(-\cos(3\cdot t) - \sin(3\cdot t))$$

Phase-Plane Trajectory for Matrix 1



Phase-Plane Trajectory for Matrix 1



" "

"----------- Matrix ", 2, " -----------"

$$A := \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$$

$$sys := \left\{ \frac{d}{dt}\, x(t) = 2\, x(t) + y(t),\ \frac{d}{dt}\, y(t) = -3\, x(t) - y(t), x(0) = 1, y(0) = 1 \right\}$$

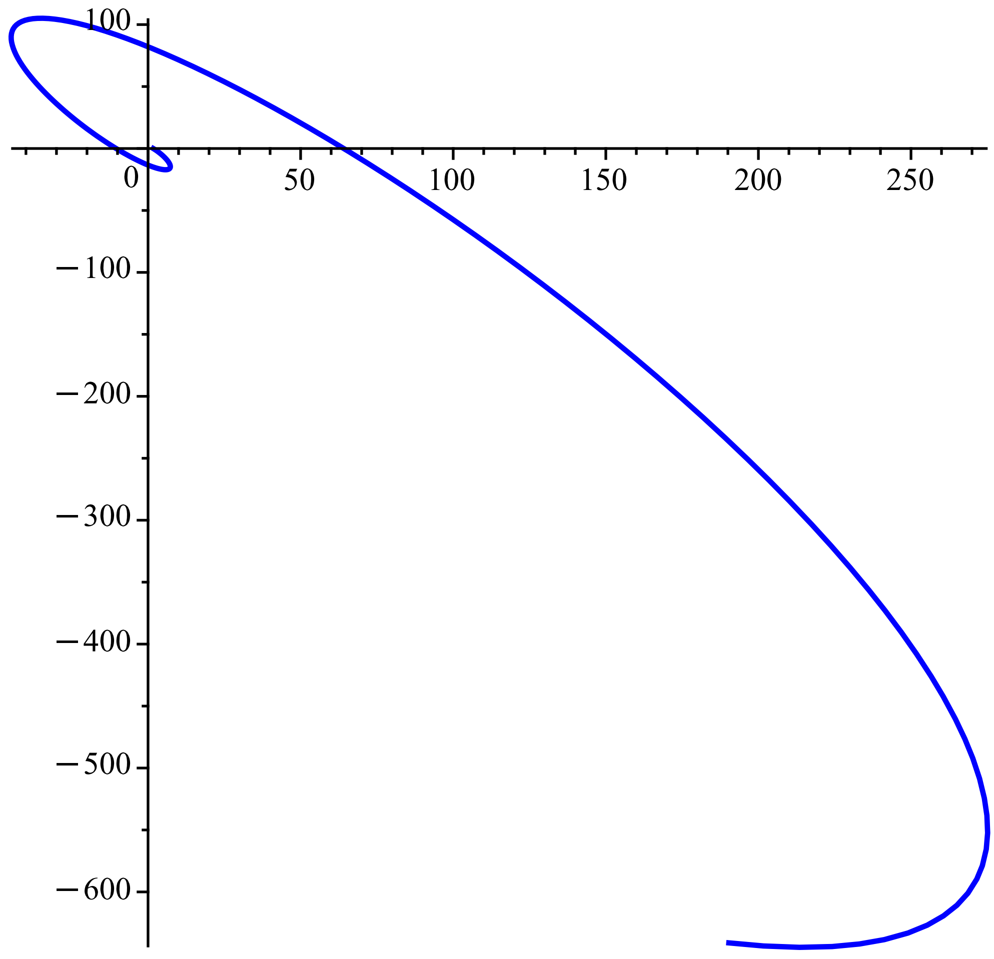
$$sol := \left\{ x(t) = e^{\frac{t}{2}} \left(\frac{5\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{3} + \cos\left(\frac{\sqrt{3}\, t}{2}\right) \right), y(t) = -\frac{e^{\frac{t}{2}} \left(6\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right) - 2\cos\left(\frac{\sqrt{3}\, t}{2}\right) \right)}{2} \right\}$$

$$\text{"Solution:"}, \left\{x(t) = e^{\frac{t}{2}} \left(\frac{5\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} + \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right), y(t) = -\frac{e^{\frac{t}{2}}\left(6\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right) - 2\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{2}\right\}$$

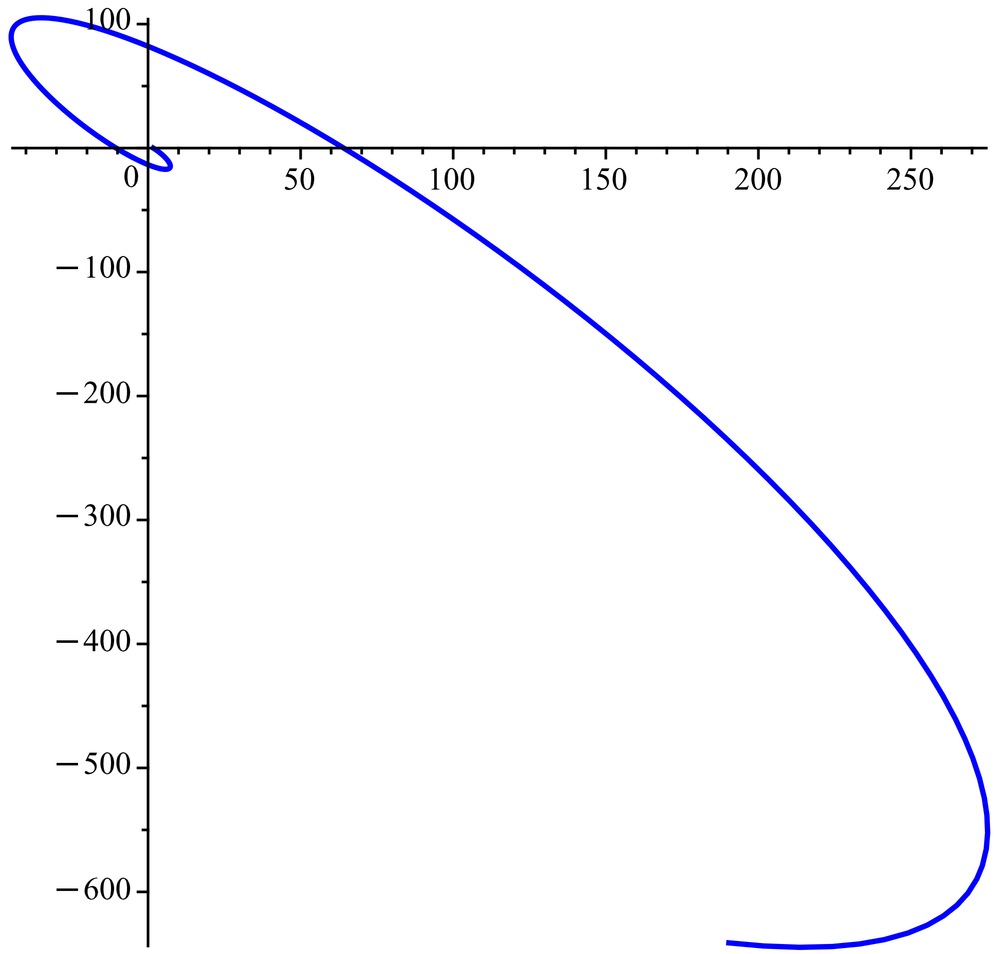
$$X := t \mapsto e^{\frac{t}{2}} \cdot \left(\frac{5\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right)}{3} + \cos\left(\frac{\sqrt{3}\cdot t}{2}\right)\right)$$

$$Y := t \mapsto -\frac{e^{\frac{t}{2}}\cdot\left(6\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right) - 2\cdot\cos\left(\frac{\sqrt{3}\cdot t}{2}\right)\right)}{2}$$

Phase-Plane Trajectory for Matrix 2



Phase-Plane Trajectory for Matrix 2



" "

"----------- Matrix ", 3, " -----------"

$$A := \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$$

$$sys := \left\{ \frac{d}{dt}\, x(t) = -2\, y(t),\ \frac{d}{dt}\, y(t) = 3\, x(t) + y(t),\ x(0) = 1,\ y(0) = 1 \right\}$$

$$sol := \left\{ x(t) = e^{\frac{t}{2}} \left(-\frac{5\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} + \cos\left(\frac{\sqrt{23}\,t}{2}\right) \right), y(t) \right.$$

$$= \frac{e^{\frac{t}{2}} \left(\frac{28 \sqrt{23} \sin\left(\frac{\sqrt{23}\ t}{2}\right)}{23} + 4 \cos\left(\frac{\sqrt{23}\ t}{2}\right) \right)}{4} \Bigg\}$$

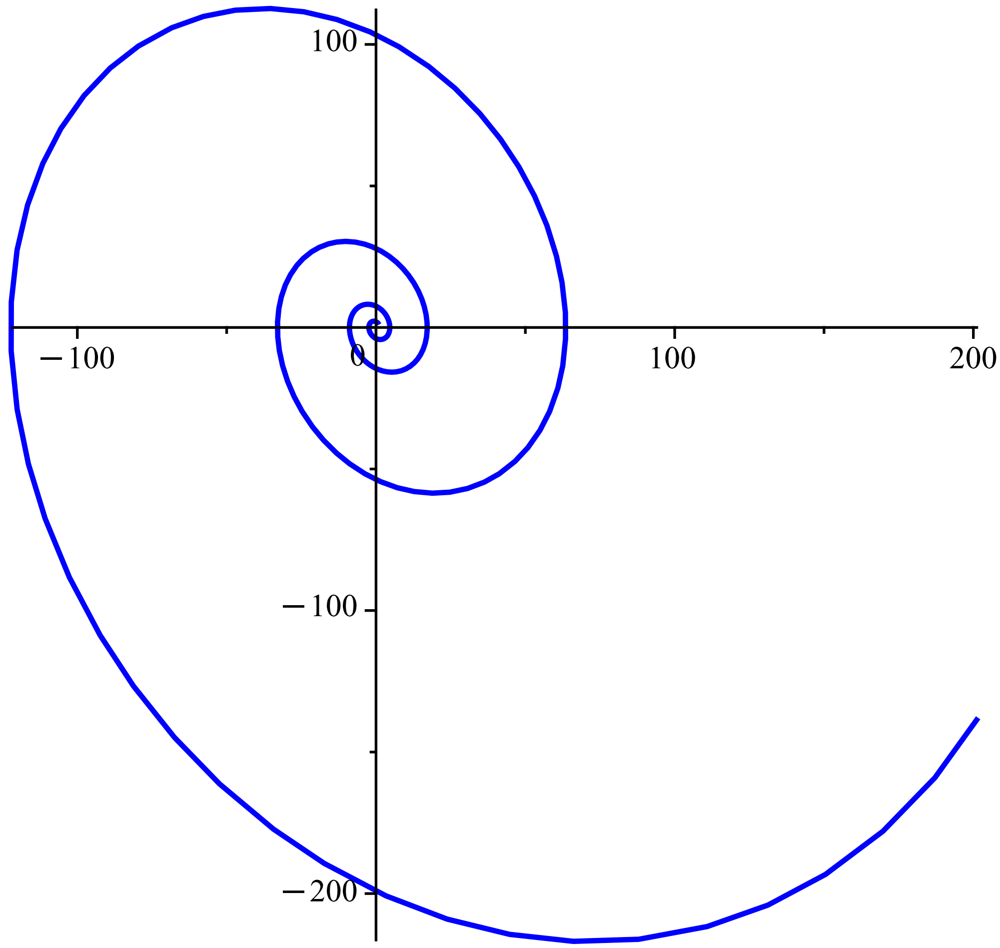
$$\text{"Solution:"}, \left\{ x(t) = e^{\frac{t}{2}} \left(-\frac{5 \sqrt{23} \sin\left(\frac{\sqrt{23}\ t}{2}\right)}{23} + \cos\left(\frac{\sqrt{23}\ t}{2}\right) \right), y(t) \right.$$

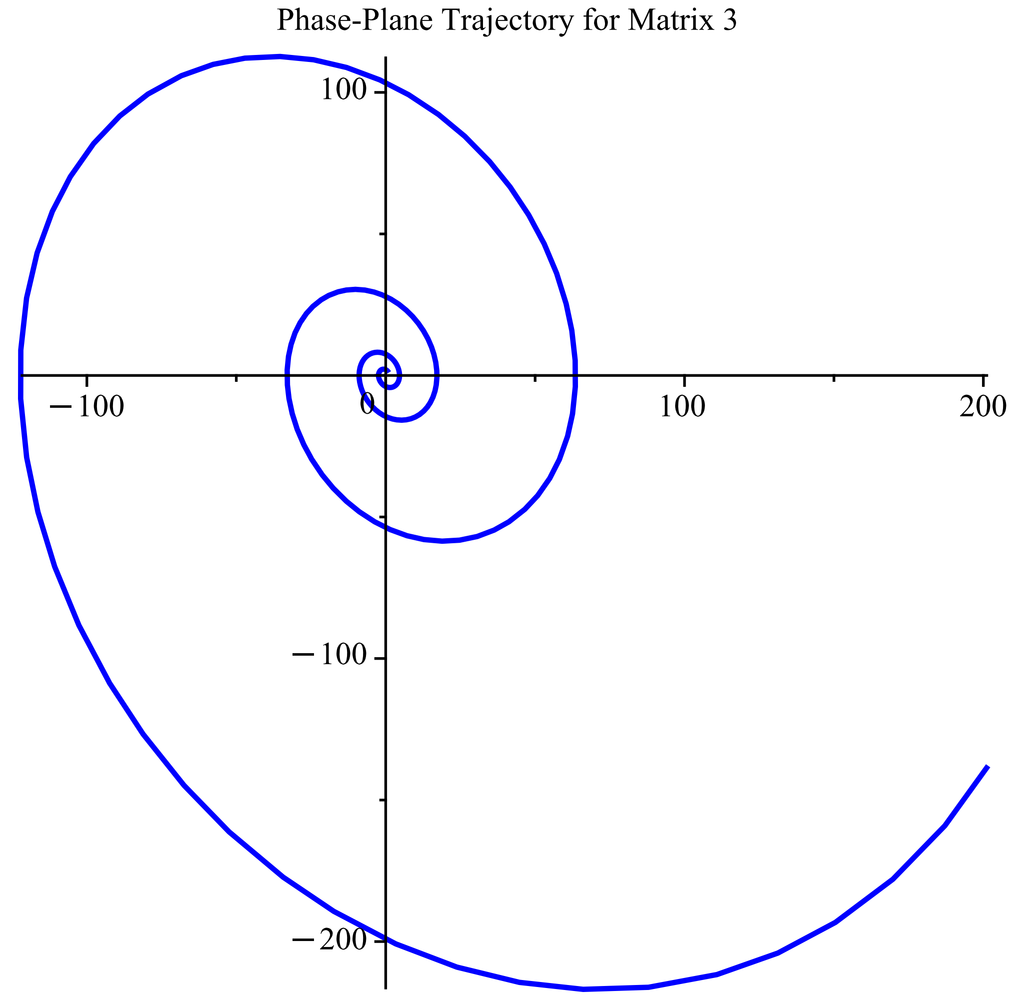
$$= \frac{e^{\frac{t}{2}} \left(\frac{28 \sqrt{23} \sin\left(\frac{\sqrt{23}\ t}{2}\right)}{23} + 4 \cos\left(\frac{\sqrt{23}\ t}{2}\right) \right)}{4} \Bigg\}$$

$$X := t \mapsto e^{\frac{t}{2}} \cdot \left(-\frac{5 \cdot \sqrt{23} \cdot \sin\left(\frac{\sqrt{23} \cdot t}{2}\right)}{23} + \cos\left(\frac{\sqrt{23} \cdot t}{2}\right) \right)$$

$$Y := t \mapsto \frac{e^{\frac{t}{2}} \cdot \left(\frac{28 \cdot \sqrt{23} \cdot \sin\left(\frac{\sqrt{23} \cdot t}{2}\right)}{23} + 4 \cdot \cos\left(\frac{\sqrt{23} \cdot t}{2}\right) \right)}{4}$$

Phase-Plane Trajectory for Matrix 3

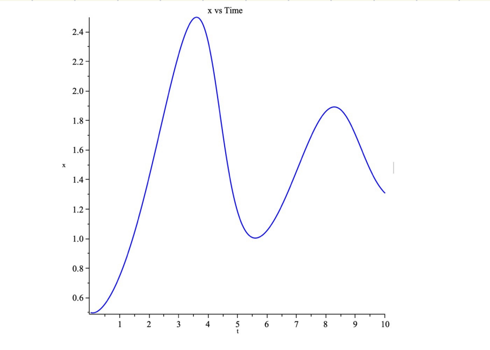


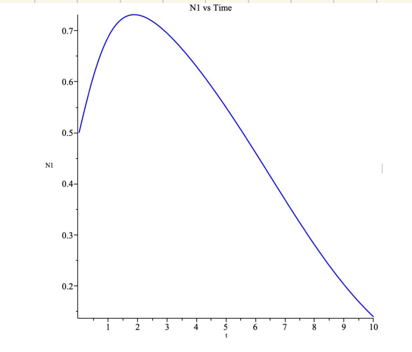


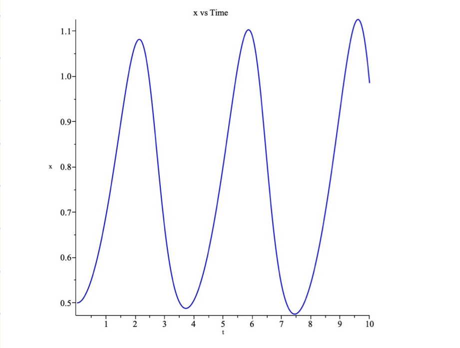
Phase-Plane Trajectory for Matrix 3
100
−100
−200
−100
0
100
200

>

Problem 4







> **read** `DMB.txt`

For a list of the Main procedures type: Help(); for help with a specific procedure type: Help(ProcedureName); for example Help(Feig);

For a list of the Continuous Dynamical Models procedures type: HelpC(); for help with a specific procedure type: Help(ProcedureName); for example Help(Feig); **(1)**

> *Help*(*Lotka*)

Lotka(r1,k1,r2,k2,b12,b21,N1,N2): The Lotka-Volterra continuous-time dynamical system, Eqs. (9a),(9b) (p. 224, section 6.3) of Edelstein-Keshet

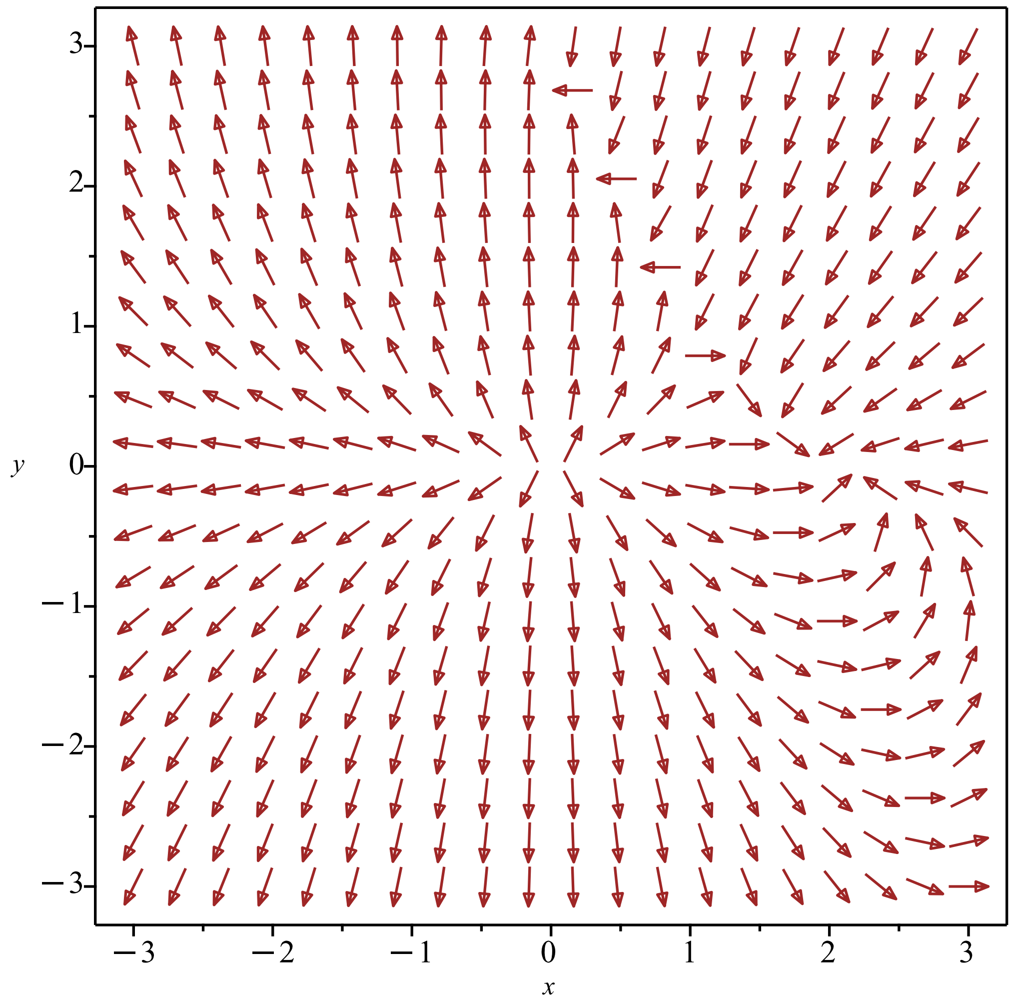
with popoluations N1, N2, and parameters r1,r2,k1,k2, b12, b21 (called there beta_12 and beta_21)

Try:

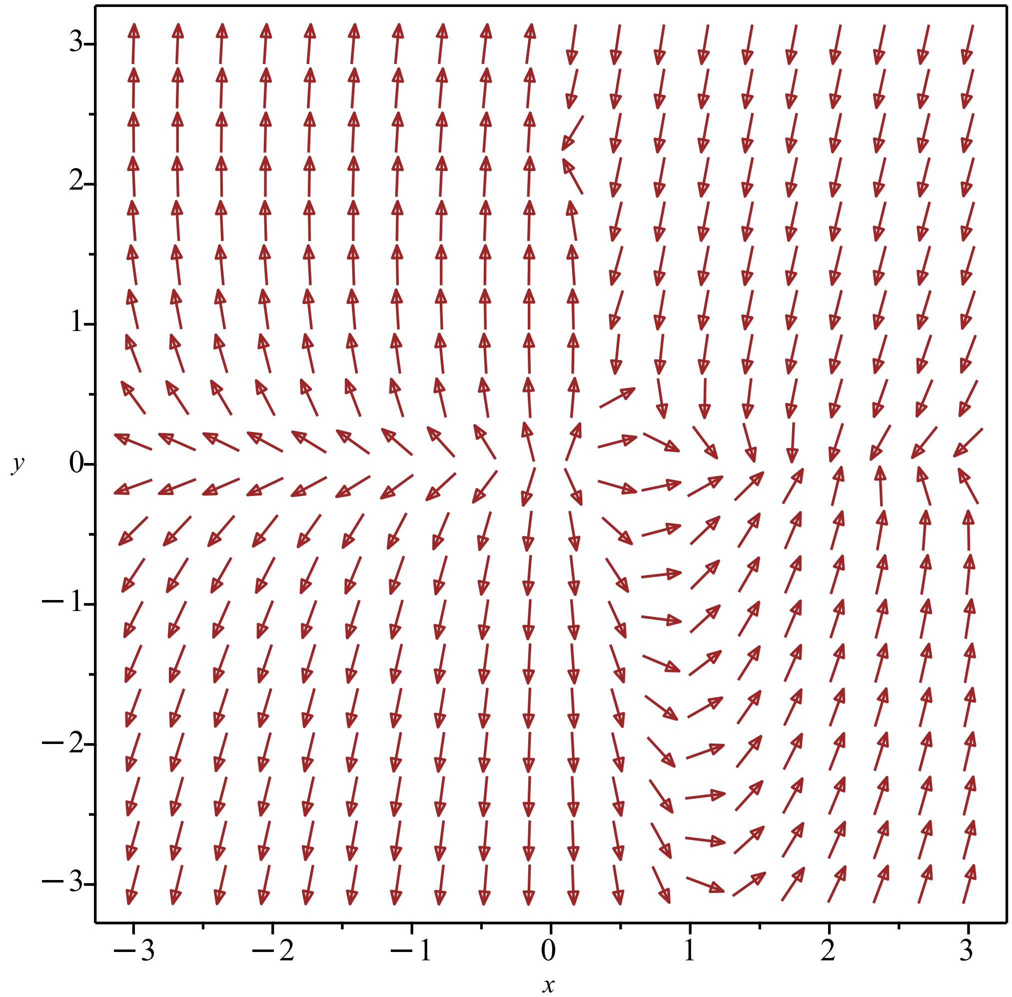
Lotka(r1,k1,r2,k2,b12,b21,N1,N2);

Lotka(1,2,2,3,1,2,N1,N2); **(2)**

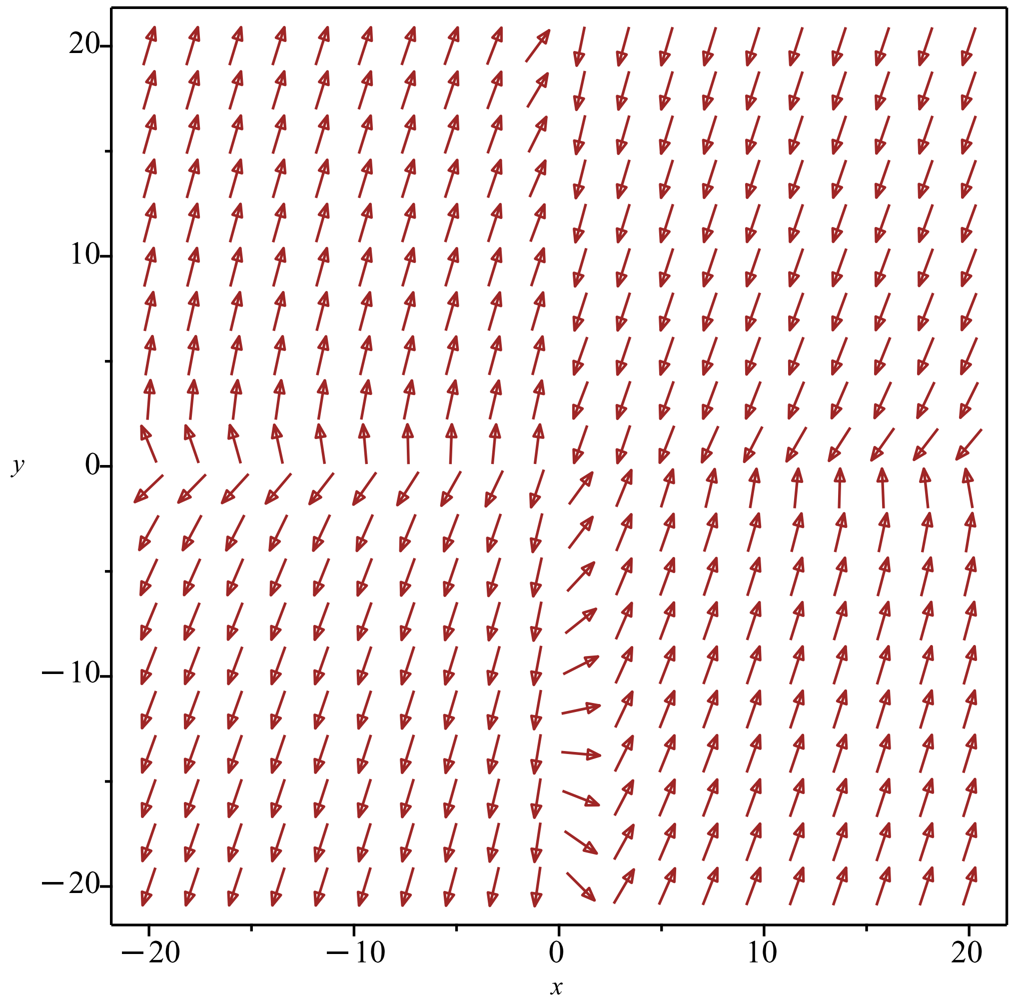
```
> with(DEtools) :

  # Define the system as a list of derivative equations
  Lotka := proc(r1, k1, r2, k2, b12, b21)
     [ diff(x(t), t) = r1*x(t)*(1 − x(t)/k1 − b12*y(t)/k1),
       diff(y(t), t) = r2*y(t)*(1 − y(t)/k2 − b21*x(t)/k2) ];
  end proc:

  # Plot the direction field
  dfieldplot(
     Lotka(1, 2, 2, 3, 1, 2),  # system of ODEs
     [x(t), y(t)],             # dependent variables
     t = 0 ..10,               # time range
     x = −3 ..3, y = −3 ..3,   # population ranges
     arrows = medium,
     axes = boxed
  );
```




```
> # Define the system as a list of derivative equations
Lotka := proc(r1, k1, r2, k2, b12, b21)
   [ diff(x(t), t) = r1*x(t) * (1 − x(t) / k1 − b12*y(t) / k1),
    diff(y(t), t) = r2*y(t) * (1 − y(t) / k2 − b21*x(t) / k2) ];
end proc:

# Plot the direction field
dfieldplot(
   Lotka(1, 2, 3, 3, 2, 5 ),  # system of ODEs
   [x(t), y(t)],             # dependent variables
   t = 0 ..10,               # time range
    x = −3 ..3, y = −3 ..3,       # population ranges
   arrows = medium,
   axes = boxed
 );
```




```
> # Define the system as a list of derivative equations
Lotka := proc(r1, k1, r2, k2, b12, b21)
   [ diff(x(t), t) = r1*x(t)*(1 - x(t)/k1 - b12*y(t)/k1),
    diff(y(t), t) = r2*y(t)*(1 - y(t)/k2 - b21*x(t)/k2) ];
end proc:

# Plot the direction field
dfieldplot(
   Lotka(1, 4, 3, 8, 10, 20), # system of ODEs
   [x(t), y(t)],          # dependent variables
   t = 0..10,             # time range
    x = -20..20, y = -20..20,      # population ranges
   arrows = medium,
   axes = boxed
 );
```



20
10
0
−10
−20
y
−20
−10
0
10
20
x

>