Dynamical Models in Biology — Homework 2

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1. **Solve** $\dfrac{dy}{dt} = \dfrac{y^3}{t+1}$ **with** $y(0) = 1$.

   Separable. For $y \neq 0$ and $t > -1$,
   $$\frac{dy}{y^3} = \frac{dt}{t+1}.$$
   Integrate:
   $$\int y^{-3}\,dy = \int (t+1)^{-1}\,dt \quad\Longrightarrow\quad -\tfrac{1}{2}y^{-2} = \ln|t+1| + C.$$
   Solve for y:
   $$y^{-2} = -2\ln(t+1) + C' \quad (t > -1).$$
   Apply $y(0) = 1$:
   $$1 = -2\ln(1) + C' \;\Rightarrow\; C' = 1.$$
   Therefore
   $$\boxed{y(t) = \frac{1}{\sqrt{1 - 2\ln(t+1)}}, \qquad -1 < t < e^{1/2} - 1}.$$

2. **Solve** $y'' - 3y' + 2y = 0$ **with** $y(0) = 2,\; y'(0) = 3$.

   Characteristic: $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$, so $r = 1, 2$. Thus
   $$y(t) = C_1e^t + C_2e^{2t}, \qquad y'(t) = C_1e^t + 2C_2e^{2t}.$$
   Initial data:
   $$\begin{cases} C_1 + C_2 = 2, \\ C_1 + 2C_2 = 3 \end{cases} \Rightarrow C_2 = 1,\; C_1 = 1.$$
   Hence
   $$\boxed{y(t) = e^t + e^{2t}}.$$

3. **Eigenvalues/eigenvectors of** $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.
   $$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 16 = 0.$$
   So
   $$(3-\lambda)^2 = -16 \;\Rightarrow\; \boxed{\lambda_{1,2} = 3 \pm 4i}.$$
   For $\lambda_1 = 3 + 4i$:
   $$A - \lambda_1 I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}, \quad -4i\,v_1 - 4\,v_2 = 0 \Rightarrow v_2 = -i\,v_1.$$

Take $v_1 = 1$:

$$\boxed{v^{(1)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}.$$

By conjugation, for $\lambda_2 = 3 - 4i$ we can take

$$\boxed{v^{(2)} = \begin{bmatrix} 1 \\ i \end{bmatrix}}.$$