Homework for Lecture 17 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Nov. 3, 2025.

Subject: hw17

with an attachment hw17FirstLast.pdf

1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 - \frac{3x}{1+y+z} \quad , \\
\frac{dy}{dt} &= 1 - \frac{3y}{1+x+z} \quad , \\
\frac{dz}{dt} &= 1 - \frac{3z}{1+x+y} \quad .
\end{aligned}$$

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 - \frac{x}{1+y+z} \quad , \\
\frac{dy}{dt} &= 1 - \frac{y}{1+x+z} \quad , \\
\frac{dz}{dt} &= 1 - \frac{z}{1+x+y} \quad .
\end{aligned}$$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$F(x): \quad \frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

$F'(x_0)<0$ stable
$F'(x_0)>0$ unstable
$F'(x_0)=0$ semistable

$0 = -(x-1)(x-4)(x-7)(x-8)$

equilibrium points : $x=1, x=4, x=7, x=8$

$$F'(x) = -(x-1)(x-4)(x-7) - (x-1)(x-7)(x-8) - (x-1)(x-4)(x-8) - (x-4)(x-7)(x-8)$$

$F'(1) = 0-0-0-(-3)(-6)(-7) > 0$ unstable

$F'(4) = 0-(3)(-3)(-4)-0-0 < 0$ stable

$F'(7) = 0-0-(6)(3)(-1) > 0$ unstable

$F'(8) = -(7)(4)(1) < 0$ stable

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$f_1: \quad \frac{dx}{dt} = 1 - \frac{3x}{1+y+z} \quad ,$$

$$f_2: \quad \frac{dy}{dt} = 1 - \frac{3y}{1+x+z} \quad ,$$

$$f_3: \quad \frac{dz}{dt} = 1 - \frac{3z}{1+x+y} \quad .$$

$1 - \frac{3x}{1+y+z} = 0$ $\quad$ $1 - \frac{3y}{1+x+z} = 0$ $\quad$ $1 - \frac{3z}{1+x+y} = 0$

$1+y+z = 3x$ $\quad$ $1+x+z = 3y$ $\quad$ $1+x+y = 3z$

equ. pt $\rightarrow (1,1,1)$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{-3}{1+y+z} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & \frac{-3}{1+x+z} & \frac{3y}{(1+y+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & \frac{-3}{1+x+y} \end{bmatrix} \overset{@(1,1,1)}{=} \begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{bmatrix}$$

real $<0 \Rightarrow$ stable

$$\begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{bmatrix}$$

found w/ maple $\rightarrow$ $\lambda = -1/3, -4/3, -4/3$

all $\lambda < 0$, therefore $(1,1,1)$ is a stable equilibrium point

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{x}{1+y+z} \quad ,$$
$$\frac{dy}{dt} = 1 - \frac{y}{1+x+z} \quad ,$$
$$\frac{dz}{dt} = 1 - \frac{z}{1+x+y} \quad .$$

$0 = 1 - \frac{x}{1+y+z}$ $\quad 1+y+z = x$

$0 = 1 - \frac{y}{1+x+z}$ $\quad 1+x+z = y$

$0 = 1 - \frac{z}{1+x+y}$ $\quad 1+x+y = z$

equilibrium pt $(-1,-1,-1)$

Check if stable

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{-1}{1+y+z} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & \frac{-1}{1+x+z} & \frac{y}{(1+y+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & \frac{-1}{1+x+y} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

find λ w/ maple $\rightarrow$ $\lambda = -1, 2, 2$

$\hookrightarrow$ unstable bc $\lambda > 0$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

$a_1 = 2$
$a_2 = 5$

$$f_1 = \frac{dN}{dt} = a_1 \frac{CN}{C+1} - N$$

$$f_2 = \frac{dC}{dt} = -\frac{CN}{C+1} - C + a_2$$

equ pts

$(0, a_2) \rightarrow \boxed{(0, 5)}$

$$\left(\frac{a_1(a_2 a_1 - a_2 - 1)}{a_1 - 1}, \frac{1}{a_1 - 1} \right) \rightarrow \left(\frac{2(10 - 5 - 1)}{1}, \frac{1}{1} \right) \rightarrow \boxed{(8, 1)}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial C} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial C} \end{bmatrix} = \begin{bmatrix} \frac{a_1 C}{C+1} - 1 & \frac{a_1 N}{(C+1)^2} \\ -\frac{C}{C+1} & -\frac{N}{(C+1)^2} - 1 \end{bmatrix}$$

$J(0,5) = \begin{bmatrix} 0 & 4 \\ -1/2 & -3 \end{bmatrix} \rightarrow \lambda = -1, -2 \rightarrow < 0$ stable

$J(8,0) = \begin{bmatrix} 2/3 & 0 \\ -5/6 & -1 \end{bmatrix} \rightarrow \lambda = 2/3, -1 \rightarrow 2/3 > 0$ unstable