Dynamical Models in Biology – HW 17

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1) Consider the one-dimensional differential equation

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8).$$

Equilibria. Equilibria satisfy $x' = 0$, so

$$-(x-1)(x-4)(x-7)(x-8) = 0.$$

Thus the equilibrium points are

$$x = 1,\ 4,\ 7,\ 8.$$

Stability via sign analysis. Let $f(x) = -(x-1)(x-4)(x-7)(x-8)$. We check the sign of $f(x)$ on each interval separated by the roots.

- For $x < 1$ (e.g. $x = 0$): $(0-1)(0-4)(0-7)(0-8)$ is the product of four negative numbers, hence positive, so $f(0) = -(+) < 0$. Solutions move to the left.
- For $1 < x < 4$ (e.g. $x = 2$): $(2-1)(2-4)(2-7)(2-8)$ has three negative factors and one positive factor, so the product is negative and $f(2) = -(-) > 0$. Solutions move to the right.
- For $4 < x < 7$ (e.g. $x = 5$): $(5-1)(5-4)(5-7)(5-8)$ is a product of two positive and two negative factors, so it is positive and $f(5) = -(+) < 0$. Solutions move to the left.
- For $7 < x < 8$ (e.g. $x = 7.5$): $(7.5-1)(7.5-4)(7.5-7)(7.5-8)$ has three positive factors and one negative factor, so it is negative and $f(7.5) = -(-) > 0$. Solutions move to the right.
- For $x > 8$ (e.g. $x = 9$): $(9-1)(9-4)(9-7)(9-8)$ is the product of four positive factors, so it is positive and $f(9) = -(+) < 0$. Solutions move to the left.

Reading off stability from the direction of arrows:

- At $x = 1$: trajectories move left on the left side and right on the right side, so they move away from $x = 1$ on both sides. Thus $x = 1$ is *unstable*.
- At $x = 4$: trajectories move right on the left side and left on the right side, so they move toward $x = 4$. Thus $x = 4$ is *asymptotically stable*.

- At $x = 7$: trajectories move left on the left side and right on the right side, so they move away from $x = 7$. Thus $x = 7$ is *unstable.*
- At $x = 8$: trajectories move right on the left side and left on the right side, so they move toward $x = 8$. Thus $x = 8$ is *asymptotically stable.*

So the stable equilibria are $x = 4$ and $x = 8$.

2) Consider the system

$$\begin{cases} \dfrac{dx}{dt} = 1 - \dfrac{3x}{1+y+z}, \\ \dfrac{dy}{dt} = 1 - \dfrac{3y}{1+x+z}, \\ \dfrac{dz}{dt} = 1 - \dfrac{3z}{1+x+y}. \end{cases}$$

Equilibrium. Set each derivative equal to zero:

$$1 - \frac{3x}{1+y+z} = 0, \qquad 1 - \frac{3y}{1+x+z} = 0, \qquad 1 - \frac{3z}{1+x+y} = 0.$$

These give the linear equations

$$3x = 1 + y + z, \quad 3y = 1 + x + z, \quad 3z = 1 + x + y.$$

Rewriting,

$$3x - y - z = 1, \quad -x + 3y - z = 1, \quad -x - y + 3z = 1.$$

By symmetry it is natural to look for $x = y = z = s$. Substituting into the first equation:

$$3s - s - s = 1 \;\Rightarrow\; s = 1.$$

Thus $(1, 1, 1)$ is an equilibrium. The coefficient matrix is invertible, so this is the unique equilibrium.

Jacobian. Let

$$f_1 = 1 - \frac{3x}{1+y+z}, \quad f_2 = 1 - \frac{3y}{1+x+z}, \quad f_3 = 1 - \frac{3z}{1+x+y}.$$

Then

$$f_{1x} = -\frac{3}{1+y+z}, \qquad f_{1y} = f_{1z} = \frac{3x}{(1+y+z)^2},$$

and by symmetry,

$$f_{2y} = -\frac{3}{1+x+z}, \quad f_{2x} = f_{2z} = \frac{3y}{(1+x+z)^2},$$

$$f_{3z} = -\frac{3}{1+x+y}, \quad f_{3x} = f_{3y} = \frac{3z}{(1+x+y)^2}.$$

At $(1,1,1)$ we have $1+x+y = 1+x+z = 1+y+z = 3$, so $x=y=z=1$. Therefore

$$J(1,1,1) = \begin{pmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{pmatrix}.$$

This matrix has diagonal entries $a=-1$ and off-diagonal entries $b=1/3$. For such a 3×3 matrix, one eigenvector is $(1,1,1)^T$ with eigenvalue

$$\lambda_1 = a+2b = -1+\frac{2}{3} = -\frac{1}{3},$$

and any vector orthogonal to $(1,1,1)$ lies in a two-dimensional eigenspace with eigenvalue

$$\lambda_2 = a-b = -1-\frac{1}{3} = -\frac{4}{3},$$

of multiplicity 2.

All eigenvalues are negative, so the equilibrium $(1,1,1)$ is an asymptotically stable node.

Thus $(1,1,1)$ is the only equilibrium, and it is asymptotically stable.

3) Consider the system

$$\begin{cases} \dfrac{dx}{dt} = 1 - \dfrac{x}{1+y+z}, \\ \dfrac{dy}{dt} = 1 - \dfrac{y}{1+x+z}, \\ \dfrac{dz}{dt} = 1 - \dfrac{z}{1+x+y}. \end{cases}$$

Equilibrium. Set derivatives to zero:

$$1-\frac{x}{1+y+z} = 0, \quad 1-\frac{y}{1+x+z} = 0, \quad 1-\frac{z}{1+x+y} = 0,$$

which give

$$x = 1+y+z, \quad y = 1+x+z, \quad z = 1+x+y.$$

Rewriting,

$$x-y-z = 1, \quad -x+y-z = 1, \quad -x-y+z = 1.$$

Again try $x=y=z=s$. The first equation then gives

$$s-s-s = 1 \;\Rightarrow\; -s = 1 \;\Rightarrow\; s = -1.$$

Thus $(-1,-1,-1)$ is an equilibrium, and it is in fact the unique equilibrium.

Jacobian. Let

$$f_1 = 1-\frac{x}{1+y+z}, \quad f_2 = 1-\frac{y}{1+x+z}, \quad f_3 = 1-\frac{z}{1+x+y}.$$

Then

$$f_{1x} = -\frac{1}{1+y+z}, \qquad f_{1y} = f_{1z} = \frac{x}{(1+y+z)^2},$$

and analogously for f_2, f_3.

At $(-1,-1,-1)$ we have $1+x+y = 1+x+z = 1+y+z = -1$, and $x = y = z = -1$. Hence

$$f_{1x} = f_{2y} = f_{3z} = -\frac{1}{-1} = 1, \qquad f_{1y} = f_{1z} = f_{2x} = f_{2z} = f_{3x} = f_{3y} = \frac{-1}{(-1)^2} = -1.$$

Thus

$$J(-1,-1,-1) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

This matrix has diagonal entries $a = 1$ and off-diagonal entries $b = -1$. Therefore

$$\lambda_1 = a + 2b = 1 - 2 = -1, \qquad \lambda_2 = a - b = 1 - (-1) = 2,$$

with λ_2 having multiplicity 2.

Since the eigenvalues have mixed signs (-1 and 2), the equilibrium $(-1,-1,-1)$ is a saddle point and hence unstable.

Thus $(-1,-1,-1)$ is the only equilibrium, and it is unstable, so this system has no stable equilibria.

4) Consider the Chemostat model

$$\begin{cases} \dfrac{dN}{dt} = \left(\dfrac{2C}{1+C} - 1\right) N, \\ \dfrac{dC}{dt} = -\dfrac{C}{1+C} N - C + 5, \end{cases}$$

with parameters $a_1 = 2$ and $a_2 = 5$.

Equilibria. At equilibrium,

$$\left(\frac{2C}{1+C} - 1\right) N = 0, \qquad -\frac{C}{1+C} N - C + 5 = 0.$$

From the first equation, either

$$N = 0 \quad \text{or} \quad \frac{2C}{1+C} - 1 = 0.$$

Case 1: $N = 0$. Then the second equation reduces to

$$-C + 5 = 0 \;\Rightarrow\; C = 5.$$

So one equilibrium is

$$(N, C) = (0, 5).$$

Case 2: $\dfrac{2C}{1+C} - 1 = 0$. Solving,

$$\frac{2C}{1+C} = 1 \ \Rightarrow\ 2C = 1 + C \ \Rightarrow\ C = 1.$$

Substituting $C = 1$ into the second equation:

$$0 = -\frac{1}{1+1}N - 1 + 5 = -\frac{1}{2}N + 4,$$

so

$$-\frac{1}{2}N + 4 = 0 \ \Rightarrow\ \frac{1}{2}N = 4 \ \Rightarrow\ N = 8.$$

Thus the second equilibrium is

$$(N, C) = (8, 1).$$

Therefore the system has two equilibria:

$$(0, 5) \quad \text{and} \quad (8, 1).$$

Jacobian and stability. Let

$$F(N, C) = \left(\frac{2C}{1+C} - 1\right)N, \qquad G(N, C) = -\frac{C}{1+C}N - C + 5.$$

Then

$$F_N = \frac{2C}{1+C} - 1, \qquad F_C = N \cdot \frac{2}{(1+C)^2},$$

$$G_N = -\frac{C}{1+C}, \qquad G_C = -\frac{N}{(1+C)^2} - 1.$$

So

$$J(N, C) = \begin{pmatrix} \dfrac{2C}{1+C} - 1 & \dfrac{2N}{(1+C)^2} \\ -\dfrac{C}{1+C} & -\dfrac{N}{(1+C)^2} - 1 \end{pmatrix}.$$

At $(N, C) = (0, 5)$ we have $1 + C = 6$, $(1+C)^2 = 36$, and $N = 0$. Then

$$F_N = \frac{2 \cdot 5}{6} - 1 = \frac{10}{6} - 1 = \frac{2}{3}, \quad F_C = 0, \quad G_N = -\frac{5}{6}, \quad G_C = -1.$$

Thus

$$J(0, 5) = \begin{pmatrix} 2/3 & 0 \\ -5/6 & -1 \end{pmatrix}.$$

Its eigenvalues are the diagonal entries,

$$\lambda_1 = \frac{2}{3} > 0, \qquad \lambda_2 = -1 < 0.$$

Since one eigenvalue is positive and one is negative, $(0, 5)$ is a saddle point and therefore unstable.

At $(N, C) = (8, 1)$ we have $1 + C = 2$, $(1 + C)^2 = 4$, and $N = 8$. Then

$$F_N = \frac{2 \cdot 1}{2} - 1 = 1 - 1 = 0, \quad F_C = \frac{2 \cdot 8}{4} = 4,$$

$$G_N = -\frac{1}{2}, \quad G_C = -\frac{8}{4} - 1 = -2 - 1 = -3.$$

So

$$J(8, 1) = \begin{pmatrix} 0 & 4 \\ -1/2 & -3 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & 4 \\ -1/2 & -3 - \lambda \end{pmatrix} = \lambda(3 + \lambda) + 2 = \lambda^2 + 3\lambda + 2.$$

Solving

$$\lambda^2 + 3\lambda + 2 = 0 \;\Rightarrow\; (\lambda + 1)(\lambda + 2) = 0,$$

we get

$$\lambda_1 = -1, \qquad \lambda_2 = -2,$$

which are both negative. Therefore $(8, 1)$ is an asymptotically stable node.

Conclusion. The equilibria are $(0, 5)$ and $(8, 1)$. The point $(0, 5)$ is a saddle (unstable), while $(8, 1)$ is asymptotically stable and is the only stable equilibrium.