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Homework, Lecture 17 DMB

1) Eq points & stable eq. 1D Dynamical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8)$$

Eq. $\Rightarrow$ $-(x-1)(x-4)(x-7)(x-8) = 0$

$x = 1, 4, 7, 8$

Eq. set of x: $\{1, 4, 7, 8\}$

$F'(x) = -(x-4)(x-7)(x-8) - (x-1)(x-7)(x-8)$
$- (x-1)(x-4)(x-8) - (x-1)(x-4)(x-7)$

when,

→ $x_0 = 1$, $F'(1) = -(-3)(-6)(-7) - (0) - (0) - (0)$

Since $F'(1) > 0 \rightarrow x_0 = 1$ is unstable

→ $x_0 = 4$, $F'(4) = 0 - (3)(-3)(-4) < 0$

$F'(4) < 0 \rightarrow x_0 = 4$ is stable

→ $x_0 = 7$, $F'(7) = -(6)(3)(-1) > 0$

$F'(7) > 0 \rightarrow x_0 = 7$ is unstable

→ $x_0 = 8$. $F'(8) = -(7)(4)(1) < 0$

$F'(8) < 0 \rightarrow x_0 = 8$ is stable

2) Stable Eq. 3D system (cont. dynamical)

Notes

Continuous Dynamical Systems:
Solve System, setting all $F(x_1, \ldots, x_n) = 0$ ⟶
Find Jacobian of Underlying Transformation ⟶
plug in Eq. points → x_0 is stable ONLY if all eigenvalues of $J(x_0)$ have neg real prt.
(If non-neg real parts and some purely imag.
⟶ Semi-stable

2) contin.
$$\begin{cases} \dfrac{dx}{dt} = 1 - \dfrac{3x}{1+y+z} \\ \dfrac{dy}{dt} = 1 - \dfrac{3y}{1+x+z} \\ \dfrac{dz}{dt} = 1 - \dfrac{3z}{1+x+y} \end{cases}$$

Eq.:

$$1 = \frac{3x}{1+y+z}, \quad 1 = \frac{3y}{1+x+z}, \quad 1 = \frac{3z}{1+x+y}$$

$$3x - y - z = 1, \quad 3y - x - z = 1, \quad 3z - x - y = 1$$

$$\left(\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 3 & 1 \end{array}\right) \xrightarrow{R_2 : R_2 + 1/3 R_1} \left(\begin{array}{ccc|c} -1 & -1 & 3 & 1 \\ -1 & 3 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & 1 \\ 0 & 4 & -4 & 0 \\ 0 & -4 & 8 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \text{eq point is } (1,1,1)$$

is it stable?

$$J_{(x,y,z)} = \begin{pmatrix} \frac{\partial}{\partial x}\left(1 - \frac{3x}{1+y+z}\right) & \frac{\partial}{\partial y}\left(1 - \frac{3x}{1+y+z}\right) & \frac{\partial}{\partial z}\left('' \right) \\ \frac{\partial}{\partial x}\left(1 - \frac{3y}{1+x+z}\right) & \frac{\partial}{\partial y}\left(1 - \frac{3y}{1+x+z}\right) & \frac{\partial}{\partial z}\left('' \right) \\ \frac{\partial}{\partial x}\left(1 - \frac{3z}{1+x+y}\right) & \frac{\partial}{\partial y}\left(1 - \frac{3z}{1+x+y}\right) & \frac{\partial}{\partial z}\left('' \right) \end{pmatrix}$$

$$J(x,y,z) = \begin{pmatrix} \frac{-3}{1+y+z} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & \frac{-3}{1+x+z} & \frac{3y}{(1+x+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & \frac{-3}{1+x+y} \end{pmatrix}$$

$$J(1,1,1) = \begin{pmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{pmatrix}$$

Using with(LinearAlgebra), I got

Eigenvalues of $J(1,1,1)$ are $\{-1/3, -4/3\}$, all have neg. real part.

$\therefore$ $x_0 = (1,1,1)$ is a stable Eq. of the 3D System.

3) Eq. & Stable Eq. 3D System:

$$\begin{cases} \frac{dx}{dt} = 1 - \frac{x}{1+y+z} \\ \frac{dy}{dt} = 1 - \frac{y}{1+x+z} \\ \frac{dz}{dt} = 1 - \frac{z}{1+x+y} \end{cases}$$

Eq. of System

$$1 = \frac{x}{1+y+z}, \quad 1 = \frac{y}{1+x+z}, \quad 1 = \frac{z}{1+x+y}$$

$$x-y-z=1, \quad y-x-z=1, \quad z-x-y=1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & -2 & 0 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \longrightarrow \text{eq. point is } (-1,-1,-1)$$

is it stable?

$$J(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x}\left(1 - \frac{x}{1+y+z}\right) & \frac{\partial}{\partial y}\left(1 - \frac{x}{1+y+z}\right) & \frac{\partial}{\partial z}\left(\text{''}\right) \\ \frac{\partial}{\partial x}\left(1 - \frac{y}{1+x+z}\right) & \frac{\partial}{\partial y}\left(1 - \frac{y}{1+x+z}\right) & \frac{\partial}{\partial z}\left(\text{''}\right) \\ \frac{\partial}{\partial x}\left(1 - \frac{z}{1+x+y}\right) & \frac{\partial}{\partial y}\left(1 - \frac{z}{1+x+y}\right) & \frac{\partial}{\partial z}\left(\text{''}\right) \end{pmatrix}$$

$$J(x,y,z) = \begin{pmatrix} \frac{-1}{1+y+z} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & \frac{-1}{1+x+z} & \frac{y}{(1+x+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & \frac{-1}{1+x+y} \end{pmatrix}$$

$$J(-1,-1,-1) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

* Do we have to show multipl?

Using with(Linear Algebra), I got

Eigenvalues of $J(-1,-1,-1)$ are $\{2, -1\}$

$\therefore$ the eq point $(-1,-1,-1)$ is Unstable

4) Chemostat Model: $a_1 = 2$; $a_2 = 5$

$$\begin{cases} \dfrac{dN}{dt} = 2\dfrac{CN}{C+1} - N \\ \dfrac{dC}{dt} = \dfrac{-CN}{C+1} - C + 5 \end{cases}$$

Eq: $\left\{ (0, a_2), \left(\dfrac{a_1(a_2 a_1 - a_2 - 1)}{a_1 - 1}, \dfrac{1}{a_1 - 1} \right) \right\}$

NOTES
- N = Bacteria population — a_1 & a_2 = parameters depend on situation
- C = Nutrients
- $(0, a_2) \rightarrow \boxed{\text{never}}$ Stable
- Other eq. $\rightarrow \boxed{\text{often}}$ stable

eq. chemostat : $\{(0,5), (8,1)\}$

STABILITY?

$(0,5) \rightarrow$ Unstable (always For All param.)

$(8,1) \rightarrow$ Next page

$$J(N,C) = \begin{pmatrix} \frac{\partial}{\partial N}\left(\frac{2CN}{C+1} - N\right) & \frac{\partial}{\partial C}\left(\frac{2CN}{C+1} - N\right) \\ \frac{\partial}{\partial N}\left(\frac{-CN}{C+1} - C + 5\right) & \frac{\partial}{\partial C}\left(\frac{-CN}{C+1} - C + 5\right) \end{pmatrix}$$

$$J(N,C) = \begin{pmatrix} \frac{2C}{C+1} - 1 & \frac{2N}{(C+1)^2} \\ \frac{-C}{C+1} & \frac{-N}{(C+1)^2} - 1 \end{pmatrix} \qquad \frac{-8}{4}$$

$$J(\overset{N}{8}, \overset{C}{1}) = \begin{pmatrix} 0 & 4 \\ -1/2 & -3 \end{pmatrix}$$

$$\det \begin{pmatrix} -\lambda & 4 \\ -1/2 & -3-\lambda \end{pmatrix} = 0 \Rightarrow (-\lambda)(-3-\lambda) - (4)(-1/2) = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0, \quad (\lambda+1)(\lambda+2) = 0$$

so $\lambda_1 = -1$, $\lambda_2 = -2$

$\therefore$ The Eq. pt. $(N,C) \to (8,1)$ is STABLE.