Homework for Lecture 17 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Nov. 3, 2025.

Subject: hw17

with an attachment hw17FirstLast.pdf

1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{3x}{1+y+z} \quad ,$$

$$\frac{dy}{dt} = 1 - \frac{3y}{1+x+z} \quad ,$$

$$\frac{dz}{dt} = 1 - \frac{3z}{1+x+y} \quad .$$

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{x}{1+y+z} \quad ,$$

$$\frac{dy}{dt} = 1 - \frac{y}{1+x+z} \quad ,$$

$$\frac{dz}{dt} = 1 - \frac{z}{1+x+y} \quad .$$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

$\boxed{\text{eq points: } x = 1, 4, 7, 8}$

$f'(x) = -(x-7)(x-4) - (x-8)(x-4) - (x-8)(x-7)(x-1) - (x-8)(x-7)(x-4)$

$f'(1) = -(-6)(-3) + 25 = 7 > 0$, not stable

$f'(4) = -36 < 0$, stable

$f'(7) = 16 > 0$, not stable

$f'(8) = -28 < 0$, stable

stable eq pts: $x = 4, 8$

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{3x}{1+y+z} \quad ,$$

$$\frac{dy}{dt} = 1 - \frac{3y}{1+x+z} \quad ,$$

$$\frac{dz}{dt} = 1 - \frac{3z}{1+x+y} \quad .$$

$0 = 1 - \frac{3x}{1+y+z}$ $\quad 0 = 1 - \frac{3y}{1+x+z}$ $\quad 0 = 1 - \frac{3z}{1+x+y}$

$1 = \frac{3x}{1+y+z}$ $\quad 1 = \frac{3y}{1+x+z}$ $\quad 1 = \frac{3z}{1+x+y}$

$1+y+z = 3x$ $\quad 1+x+z = 3y$ $\quad 1+x+y = 3z$

$$\begin{array}{r} 1+y+z = 3x \\ -\ 1+x+z = 3y \\ \hline y-x = 3x-3y \\ y+3y = 3x+x \\ 4y = 4x \\ y = x \end{array} \qquad \begin{array}{r} 1+x+z = 3y \\ -\ 1+x+y = 3z \\ \hline z-y = 3y-3z \\ -4y = -4z \\ y = z \end{array}$$

$x = y = z = \alpha$

$1 - \frac{3\alpha}{1+2\alpha} = 0$

$1 = \frac{3\alpha}{1+2\alpha}$

$1 + 2\alpha = 3\alpha$

$\alpha = 1 \Rightarrow (x, y, z) = (1, 1, 1)$

$f(x) = 1 - \frac{3x}{1+y+z}$, $f(x) = 1 - \frac{3y}{1+x+z}$, $f(x) = 1 - \frac{3z}{1+x+y}$

$$J = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{-3}{1+y+z} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & \frac{-3}{(1+x+z)} & \frac{3y}{(1+x+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & \frac{-3}{1+x+y} \end{bmatrix}$$

$$J(1,1,1) = \begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}$$

eigenval from maple: $\lambda = -\frac{1}{3}, -\frac{4}{3}, -\frac{4}{3}$

$\hookrightarrow$ all neg $\therefore$ stable eq pt $(1,1,1)$

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{x}{1+y+z} \quad ,$$

$$\frac{dy}{dt} = 1 - \frac{y}{1+x+z} \quad ,$$

$$\frac{dz}{dt} = 1 - \frac{z}{1+x+y} \quad .$$

$0 = 1 - \frac{x}{1+y+z}$ $\qquad$ $0 = 1 - \frac{y}{1+x+z}$ $\qquad$ $0 = 1 - \frac{z}{1+x+y}$

$1+y+z = x$ $\qquad$ $1+x+z = y$ $\qquad$ $1+x+y = z$

eq. pts : $(-1,-1,-1)$

$$J = \begin{bmatrix} \frac{-1}{1+y+z} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & \frac{-1}{1+x+z} & \frac{y}{(1+x+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & \frac{-1}{1+x+y} \end{bmatrix}$$

$$J(-1,-1,-1) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A - \lambda I : \begin{bmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{bmatrix} \longrightarrow (-1)^{1+1}(1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} + (-1)^{1+2}(-1)\det\begin{bmatrix} -1 & -1 \\ -1 & 1-\lambda \end{bmatrix} + (-1)^{1+3}(-1)\det\begin{bmatrix} -1 & 1-\lambda \\ -1 & -1 \end{bmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] + (2-\lambda) + (-1)(2-\lambda) = 0$$

$$= -\lambda^3 + 3\lambda^2 - 4 = 0$$

$$-(\lambda+1)(\lambda^2 - 4\lambda + 4) = 0$$

$$-(\lambda+1)(\lambda-2)^2 = 0$$

$\lambda = -1, \underbrace{2, 3}_{\text{not neg so pt is unstable}}$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

N = bacteria

C = food

a_1, a_2 = bio param
$\;\;2\;\;\;5$

$\frac{dN}{dt} = a_1 \frac{CN}{C+1} - N \longrightarrow 0 = a_1 \frac{CN}{C+1} - N \longrightarrow a_1 \frac{CN}{C+1} = N$

$\frac{dC}{dt} = -\frac{CN}{C+1} - C + a_2 \longrightarrow 0 = -\frac{CN}{C+1} - C + a_2 \longrightarrow a_2 = \frac{CN}{C+1} + C$

} if $N=0$, $C = a_2$

eq pt: $(0,5)$

$\left(\frac{a_1(a_2 a_1 - a_2 - 1)}{a_1 - 1}, \frac{1}{a_1 - 1}\right)$; $a_1 = 2$, $a_2 = 5$

eq pt: $(8,1)$

$J(8,1) = \begin{bmatrix} 0 & 4 \\ -1/2 & -3 \end{bmatrix}$

$(-\lambda)(-3-\lambda) - 2 = 0$

$3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 + 3\lambda - 2 = 0$

$= \frac{-3 \pm \sqrt{9 - 4(1)(-2)}}{2}$

$= \frac{-3 \pm \sqrt{9+8}}{2}$

$= \frac{-3 \pm \sqrt{17}}{2} \Rightarrow \lambda = \frac{-3+\sqrt{17}}{2}, \frac{-3-\sqrt{17}}{2}$

$(0.56)\quad(-3.56)$

unstable b/c $\frac{-3+\sqrt{17}}{2} > 0$

$J(0,5) = \begin{bmatrix} \frac{Ca_1 - C - 1}{C+1} & \frac{a_1 N}{(C+1)^2} \\ -\frac{C}{C+1} & -\frac{N}{(C+1)^2} - 1 \end{bmatrix}$

$J(0,5) = \begin{bmatrix} \frac{2}{3} & 0 \\ -\frac{5}{6} & -1 \end{bmatrix}$

$(\frac{2}{3} - \lambda)(-1-\lambda) - 0 = 0$

$\lambda = \frac{2}{3}, -1$

unstable b/c $2/3 > 0$