MATH336 - HW #17

1. $\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8)$

equilibria where $\frac{dx}{dt} = 0$, so equilibria values are $x = \{1, 4, 7, 8\}$

ignoring the first negative sign for now, $f'(x) = (x-4)(x-7)(x-8) + (x-1)(x-4)(x-7) + (x-1)(x-4)(x-8) + (x-1)(x-7)(x-8$

- at $x=1$, $f'(x)$ is negative, but multiplying by -1 makes $f'(x)$ positive
- at $x=4$, $f'(x)$ is positive, but multiplying by -1 makes $f'(x)$ negative
- at $x=7$, $f'(x)$ is negative, but multiplying by -1 makes $f'(x)$ positive
- at $x=8$, $f'(x)$ is positive, but multiplying by -1 makes $f'(x)$ negative

these 2 are stable since $f'(x) < 0$

$$\boxed{\begin{array}{l}\text{equilibria at } x = \{1,4,7,8\} \\ \text{stable equilibria at } x = 4, 8\end{array}}$$

2. $$\begin{cases} \frac{dx}{dt} = 1 - \frac{3x}{1+y+z} \\ \frac{dy}{dt} = 1 - \frac{3y}{1+x+z} \\ \frac{dz}{dt} = 1 - \frac{3z}{1+x+y} \end{cases}$$

all 3 eqns. have some structure, so I'm expecting some sort of symmetry (ie. at the equilibrium, $x=y=z$).

$\frac{dx}{dt} = 1 - \frac{3x}{1+y+z}$; $0 = 1 - \frac{3x}{1+y+z}$; $1 = \frac{3x}{1+y+z}$, but if $x=y=z$, then $1 = \frac{3x}{1+2x}$; $3x = 1+2x$; $x=1$

equilibrium at $(1,1,1)$

$$J = \begin{bmatrix} \frac{-3}{(1+y+z)} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & \frac{-3}{(1+x+z)} & \frac{3y}{(1+x+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & \frac{-3}{(1+x+y)} \end{bmatrix} ; J(1,1,1) = \begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{bmatrix}$$

According to Maple, eigenvalues are $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = \frac{-4}{3}$ (multiplicity of 2). Since all $\lambda < 0$, equilibrium is stable.

$$\boxed{\text{stable equilibrium at } (1,1,1)}$$

3. $\begin{cases} \frac{dx}{dt} = 1 - \frac{x}{1+y+z} \\ \frac{dy}{dt} = 1 - \frac{y}{1+x+z} \\ \frac{dz}{dt} = 1 - \frac{z}{1+x+y} \end{cases}$ I'll use a similar symmetry argument as problem #2 (i.e., at equilibrium, $x=y=z$)

$0 = 1 - \frac{x}{1+y+z}$; $1 = \frac{x}{1+y+z}$, but $x=y=z$, so $1 = \frac{x}{1+2x}$; $x = 1+2x$; $x=-1$

equilibrium at $(-1,-1,-1)$

$$J = \begin{bmatrix} \frac{-1}{(1+y+z)} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & \frac{-1}{(1+x+z)} & \frac{y}{(1+x+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & \frac{-1}{(1+x+y)} \end{bmatrix}; J(-1,-1,-1) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

According to Maple, eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$ (multiplicity of 2). Since there is one negative λ and one positive λ, the equilibrium is unstable.

$\boxed{\text{equilibrium at } (-1,-1,-1) \text{ which is unstable}}$

4. Equilibria of chemostat is $\overbrace{(0, a_2)}^{\text{usually unstable}}$ and $\overbrace{\left(\frac{a_1(a_2 a_1 - a_2 - 1)}{a_1 - 1}, \frac{1}{a_1 - 1}\right)}^{\text{usually stable}}$

$a_1 = 2, a_2 = 5$

equilibria at $(0,5)$ and $(8,1)$

$\boxed{\begin{array}{l}\text{unstable equilibrium at } (0,5) \\ \text{stable equilibrium at } (8,1)\end{array}}$