Homework for Lecture 17 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

ShaloshBEkhad@gmail.com

by 8:00pm Monday, Nov. 3, 2025.

Subject: hw17

with an attachment hw17FirstLast.pdf

1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 - \frac{3x}{1+y+z} \quad , \\
\frac{dy}{dt} &= 1 - \frac{3y}{1+x+z} \quad , \\
\frac{dz}{dt} &= 1 - \frac{3z}{1+x+y} \quad .
\end{aligned}$$

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 - \frac{x}{1+y+z} \quad , \\
\frac{dy}{dt} &= 1 - \frac{y}{1+x+z} \quad , \\
\frac{dz}{dt} &= 1 - \frac{z}{1+x+y} \quad .
\end{aligned}$$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

Daniyal Chaudhry HW17

1. $\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8)$ clear zeroes at $x=1,4,7,8$: these are equilibria

using the generalized product rule: $df/dx = -1[(x-1)(x-4)(x-7) + (x-1)(x-4)(x-8) + (x-1)(x-7)(x-8) + (x-4)(x-7)(x-8)]$

Using negative or positive: $x=1$: positive, unstable. $x=4$: negative, stable. $x=7$: positive, unstable. $x=8$: negative, stable

2. $\frac{dx}{dt} = 1 - \frac{3x}{1+y+z}$ $\frac{dy}{dt} = 1 - \frac{3y}{1+x+z}$ $\frac{dz}{dt} = 1 - \frac{3z}{1+x+y}$

the only eq. point is when $\frac{3x}{1+y+z} = \frac{3y}{1+x+z} = \frac{3z}{1+x+y} = 1$, which happens for $x=y=z=1$.

$J \Rightarrow \frac{df_1}{dx} = \frac{3}{1+y+z}$ $\frac{df_1}{dy} = \frac{3x}{(1+y+z)^2}$ $\frac{df_1}{dz} = \frac{3x}{(1+y+z)^2}$

same process for others:

$$J = \begin{bmatrix} -\frac{3}{1+y+z} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & -\frac{3}{1+x+z} & \frac{3y}{(1+x+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & -\frac{3}{1+x+y} \end{bmatrix} \quad J(1,1,1) = \begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & -1 \end{bmatrix}$$

$\lambda = -1/3, -4/3$: all negative.

$(1,1,1)$ is stable

3. $\frac{dx}{dt} = 1 - \frac{x}{1+y+z}$ $\frac{dy}{dt} = 1 - \frac{y}{1+x+z}$ $\frac{dz}{dt} = \frac{z}{1+x+y}$

We need all fractions $= 1$, as before. We are forced to have $x=y=z=$"a". Let $\frac{a}{1+2a} = 1$.

$a = 1+2a \rightarrow a=-1$. Test: $1 - \frac{(-1)}{1-2} = 0$ ✓ so $(-1,-1,-1)$ is the only eq. point.

$J \Rightarrow \frac{df_1}{dx} = \frac{1}{1+y+z}$ $\frac{df_1}{dy} = \frac{x}{(1+y+z)^2}$ $\frac{df_1}{dz} = \frac{x}{(1+y+z)^2}$

same process for the rest

$$J = \begin{bmatrix} -\frac{1}{1+y+z} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & -\frac{1}{1+x+z} & \frac{y}{(1+x+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & -\frac{1}{1+x+y} \end{bmatrix} \quad J(-1,-1,-1): \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$\lambda = -1, 2$, so $(-1,-1,-1)$ is not stable.

4. chemostat where $q_1 = 2$, $q_2 = 5$:

$\frac{dN}{dt} = 2\frac{CN}{C+1} - N$ $\frac{dC}{dt} = -\frac{CN}{C+1} - C + 5$ has eq points $(0,5)$ and $\left(\frac{2(10-5-1)}{2-1}, \frac{1}{2-1}\right) = (8,1)$

$(0,5)$ is not stable according to the notes.

$$J = \begin{bmatrix} \frac{2C}{C+1} - 1 & \frac{2N}{(C+1)^2} \\ -\frac{C}{C+1} & -\frac{N}{(C+1)^2} \end{bmatrix} \rightarrow J(8,1) = \begin{bmatrix} 0 & 4 \\ -1/2 & -3 \end{bmatrix}$$

$\lambda = -1, -2$ stable at $(8,1)$