1. Find all the equilibrium points and stable equilibrium points of the following one-dimensional dynmaical system

$$\frac{dx}{dt} = -(x-1)(x-4)(x-7)(x-8) \quad .$$

$\boxed{\text{eq pts: } x = \{1,4,7,8\}}$

$f'(x) = -(x-7)(x-4) - (x-8)(x-4)(x-1) - (x-8)(x-7)(x-1) - (x-8)(x-7)(x-4)$

$f'(1) = -18 + 25 = 7 > 0 \rightarrow$ not stable

$f'(4) = -36 < 0 \rightarrow$ stable

$f'(7) = 16 > 0 \rightarrow$ not stable

$f'(8) = -28 < 0 \rightarrow$ stable

$\boxed{\text{Stable equilibrium pts } x = \{4, 8\}}$

2. (You can use Maple for the eigenvalues, but not for the Jacobian)

Find all stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt} = 1 - \frac{3x}{1+y+z} \quad ,$$

$$\frac{dy}{dt} = 1 - \frac{3y}{1+x+z} \quad ,$$

$$\frac{dz}{dt} = 1 - \frac{3z}{1+x+y} \quad .$$

eq. pts. $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$

$1 - \frac{3x}{1+y+z} = 0 \rightarrow 3x = 1+y+z$

$x = \frac{1+y+z}{3}$

$1 - \frac{3y}{1+x+z} = 0 \rightarrow y = \frac{1+x+z}{3}$

$1 - \frac{3z}{1+x+y} = 0 \rightarrow z = \frac{1+x+y}{3}$

use $x=y=z=s$ (symmetry)

$1 - \frac{3s}{1+2s} = 0$

s

$1 = \frac{3s}{1+2s}$

$1 + 2s = 3s$

$s = 1 \rightarrow (x,y,z) = (1,1,1)$

an eq pt $(1,1,1)$

say $x = y = a$, $z = b$

$\frac{dx}{dt} \Rightarrow 1 - \frac{3a}{1+a+b} = 0$

$3a = 1+a+b$

$2a = 1+b$

$b = 2a-1$

$\frac{dz}{dt} \Rightarrow 1 - \frac{3b}{1+2a} = 0$

$\uparrow a=1, b=1$

$3b = 1+2a$

$3(2a-1) = 1+2a$

since $a=1$, $b=1$

$\boxed{\text{the only eq. pt is } (1,1,1)}$

$$6a-3=1+2a$$
$$4a=4$$
$$a=1$$

Jacobian

Use $f(x)=1-\frac{3x}{1+y+z}$ } all others follow same pattern!

$$\frac{\partial f}{\partial x}=\frac{-3}{1+y+z}$$
$$\frac{\partial f}{\partial y}=\frac{3x}{(1+y+z)^2}$$
$$\frac{\partial f}{\partial z}=\frac{3x}{(1+y+z)^2}$$

$$J(x,y,z)=\begin{bmatrix} \frac{-3}{1+y+z} & \frac{3x}{(1+y+z)^2} & \frac{3x}{(1+y+z)^2} \\ \frac{3y}{(1+x+z)^2} & \frac{-3}{1+y+z} & \frac{3y}{(1+x+z)^2} \\ \frac{3z}{(1+x+y)^2} & \frac{3z}{(1+x+y)^2} & \frac{-3}{1+x+y} \end{bmatrix}$$

$$J(1,1,1)=\begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}$$

Eigenvalues
$\det(A-\lambda I)=0$

used maple. $\lambda=\{-\frac{1}{3}, -\frac{4}{3}, -\frac{4}{3}\}$

↳ all negative → stable eq. pt (1,1,1)

3. Find all the the equilibria and stable equilibria of the 3-dimensinal dynamical system:

$$\frac{dx}{dt}=1-\frac{x}{1+y+z} \quad ,$$
$$\frac{dy}{dt}=1-\frac{y}{1+x+z} \quad ,$$
$$\frac{dz}{dt}=1-\frac{z}{1+x+y} \quad .$$

set $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0$

$$1-\frac{x}{1+y+z}=0 \rightarrow \quad x=1+y+z$$
$$1-\frac{y}{1+x+z}=0 \rightarrow \quad y=1+x+z$$
$$1-\frac{z}{1+x+y}=0 \rightarrow \quad z=1+x+y$$

$$x=1-1-1 \Rightarrow x=-1$$

$$\cancel{y}=1+1+\cancel{y}+z+z \rightarrow -2=2z \rightarrow z=-1$$
$$\cancel{z}=1+1+y+\cancel{z}+y \rightarrow -2=2y \rightarrow y=-1$$

$\boxed{\text{eq. pt @ } (-1,-1,-1)}$

$$f(x)=1-\frac{x}{1+y+z}$$

$\frac{\partial f}{\partial x}=\frac{-1}{1+y+z}$ $\quad \frac{\partial f}{\partial y}=\frac{x}{(1+y+z)^2}$ $\quad \frac{\partial f}{\partial z}=\frac{x}{(1+y+z)^2}$ } the others follow the same pattern

$$J = \begin{bmatrix} \frac{-1}{1+y+z} & \frac{x}{(1+y+z)^2} & \frac{x}{(1+y+z)^2} \\ \frac{y}{(1+x+z)^2} & \frac{-1}{1+x+z} & \frac{y}{(1+x+z)^2} \\ \frac{z}{(1+x+y)^2} & \frac{z}{(1+x+y)^2} & \frac{-1}{1+x+y} \end{bmatrix}$$

$$J(-1,-1,-1) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = 0$$

$$(-1)^{1+1}(1-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} + (-1)^{1+2}(-1)\det\begin{bmatrix} -1 & -1 \\ -1 & 1-\lambda \end{bmatrix}$$

$$+ (-1)^{1+3}(-1)\det\begin{bmatrix} -1 & 1-\lambda \\ -1 & -1 \end{bmatrix}$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] + (2-\lambda) + (-1)(2-\lambda) = 0$$

$$-\lambda^3 + 3\lambda^2 - 4 = 0$$
$$-(\lambda+1)(\lambda^2 - 4\lambda + 4) = 0$$
$$-(\lambda+1)(\lambda-2)^2 = 0$$
$$\lambda = \{-1, 2, 2\}$$

$\boxed{\text{Since the eigenvalues are } \lambda = \{-1,2,2\} \text{ and } 2,2 \text{ aren't negative, this pt is unstable}}$

4. What are the equilibria, and stable equilibria of the Chemostat model with parameters $a_1 = 2$ and $a_2 = 5$?

N: bacterial population
C: nutrient (food)
a_1, a_2: biological parameters

$$\frac{dN}{dt} = \overset{2}{a_1}\frac{CN}{C+1} - N$$

$$\frac{dC}{dt} = -\frac{CN}{C+1} - C + \overset{5}{a_2}$$

set $\frac{dN}{dt} = \frac{dC}{dt} = 0$

$$\frac{dN}{dt} = a_1\frac{CN}{C+1} - N$$

$$\frac{dC}{dt} = -\frac{CN}{C+1} - C + a_2$$

$$a_1\frac{CN}{C+1} - N = 0$$

$$a_1\frac{CN}{C+1} = N \quad \text{if } N = 0$$

$$\frac{CN}{C+1} + C = a_2 \qquad C = a_2$$

eq pt @ $(0,5)$

an eq. pt → $\left(\frac{a_1(a_2a_1 - a_2 - 1)}{a_1 - 1}, \frac{1}{a_1 - 1}\right)$

$$\left(\frac{2(5(2) - 5 - 1)}{2-1}, \frac{1}{2-1}\right)$$

eq pt @ $(8,1)$

Jacobian

$$J(N,C) = \begin{bmatrix} \frac{Ca_1 - C - 1}{C+1} & \frac{a_1 N}{(C+1)^2} \\ -\frac{C}{C+1} & -\frac{N}{(C+1)^2} - 1 \end{bmatrix}$$

$$J(8,1) = \begin{bmatrix} 0 & 4 \\ \frac{-1}{2} & -3 \end{bmatrix}$$

$$(-\lambda)(-3-\lambda) + 2 = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda+1)(\lambda-2) = 0$$

$\lambda = \{-1, 2\}$ → unstable

$$J(0,5) = \begin{bmatrix} \frac{2}{3} & 0 \\ \frac{-5}{6} & -1 \end{bmatrix}$$

$$\left(\tfrac{2}{3} - \lambda\right)(-1-\lambda) + 0 = 0$$

$\lambda = \{\frac{2}{3}, -1\}$

unstable
b/c $\frac{2}{3} > 0$