

hw 13 Zachariah Ali

1. a) $x_1 = \frac{x_1}{3+x_2} \Rightarrow x_1 - \frac{x_1}{3+x_2} = 0 \Rightarrow x_1 \left(1 - \frac{1}{3+x_2}\right) = 0$

$x_2 = \frac{x_2}{2+x_1} \Rightarrow x_2 \left(1 - \frac{1}{2+x_1}\right) = 0$

Note we have a transformation $(x_1, x_2) \rightarrow \left(\frac{x_1}{3+x_2}, \frac{x_2}{2+x_1}\right)$

Either $x_1 = 0$ or $1 = \frac{1}{3+x_2}$. $x_1 = 0 \Rightarrow x_2 \left(1 - \frac{1}{2+0}\right) = 0 \Rightarrow x_2 = 0$

And either $x_2 = 0$ or $1 = \frac{1}{2+x_1}$. $x_2 = 0 \Rightarrow x_1 \left(1 - \frac{1}{3+0}\right) = 0 \Rightarrow x_1 = 0$

So $(0,0)$ is a steady state.

Also if $1 = \frac{1}{3+x_2}$ and $1 = \frac{1}{2+x_1}$, then $x_2 = -2$ and $x_1 = -1$

So $(-1, -2)$ is a steady state.

Check $x_2 = -2 \Rightarrow x_1$

b) $f_1 = \frac{x_1}{3+x_2}$ $f_2 = \frac{x_2}{2+x_1}$

$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3+x_2} & \frac{-x_1}{(3+x_2)^2} \\ \frac{-x_2}{(2+x_1)^2} & \frac{1}{2+x_1} \end{pmatrix}$

At $(0,0)$. $J = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

$\det(J - I\lambda) = \left(\frac{1}{3} - \lambda\right) \left(\frac{1}{2} - \lambda\right) = 0$

Eigenvalues $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{2}$. $|\lambda_i| < 1 \forall i \in \mathbb{N}$.

So $(0,0)$ is stable.

At $(-1, -2)$ $J = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$$\det(5 - \lambda I) = (1 - \lambda)^2 - 1 \times 2 = 0$$

$$= (1 - \lambda)^2 - 2$$

$$\Rightarrow 1 - \lambda = \pm \sqrt{2}$$

$$\lambda = 1 \pm \sqrt{2}$$

$$\lambda_1 = 1, \lambda_2 = 3$$

$|1 + \sqrt{2}| > 1$ so $(-1, -2)$ is unstable.

$$2. a) (x_1, x_2) \rightarrow \left(\frac{x_1}{3x_1 + 5x_2}, \frac{x_2}{2x_1 + 7x_2} \right)$$

$$x_1 = \frac{x_1}{3x_1 + 5x_2}$$

$$x_2 = \frac{x_2}{2x_1 + 7x_2}$$

$$\Rightarrow x_1 \left(1 - \frac{1}{3x_1 + 5x_2} \right) = 0$$

$$\Rightarrow x_2 \left(1 - \frac{1}{2x_1 + 7x_2} \right) = 0$$

$$\Rightarrow x_1 = 0 \text{ or } 1 = \frac{1}{3x_1 + 5x_2}$$

$$\Rightarrow x_2 = 0 \text{ or } 1 = \frac{1}{2x_1 + 7x_2}$$

$(x_1, x_2) = (0, 0)$ cannot be as the transformation is undefined.

If $x_1 = 0$ then $1 - \frac{1}{2x_0 + 7x_2} = 0$

so $1 = \frac{1}{7x_2}$ and $x_2 = \frac{1}{7}$

So we get $(0, \frac{1}{7})$.

If $x_2 = 0$ then $1 = \frac{1}{3x_1 + 0} \Rightarrow x_1 = \frac{1}{3}$

and $(\frac{1}{3}, 0)$ is a steady state.

If $1 = \frac{1}{3x_1 + 5x_2}$ and $1 = \frac{1}{2x_1 + 7x_2}$

Then $3x_1 + 5x_2 = 1$ and $2x_1 + 7x_2 = 1$

$$\Rightarrow x_1 = \frac{1 - 5x_2}{3} \Rightarrow 7x_1 + \frac{7}{3} - \frac{10}{3}x_2 = 1$$

$$\Rightarrow x_2 = \frac{1}{11} \Rightarrow x_1 = \frac{2}{11}$$

$s_0 \left(\frac{2}{11}, \frac{1}{11} \right)$ is a steady state

b) $s_1 = \frac{x_1}{3x_1 + 5x_2}$ $s_2 = \frac{x_2}{2x_1 + 7x_2}$

$$J = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} \\ \frac{\partial s_2}{\partial x_1} & \frac{\partial s_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{1 \times (3x_1 + 5x_2) - x_1(3)}{(3x_1 + 5x_2)^2} & \frac{0 - x_1(5)}{(3x_1 + 5x_2)^2} \\ \frac{0 - x_2(-2)}{(2x_1 + 7x_2)^2} & \frac{1 \times (2x_1 + 7x_2) - x_2(7)}{(2x_1 + 7x_2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-5x_1}{(3x_1 + 5x_2)^2} & \frac{-5x_1}{(3x_1 + 5x_2)^2} \\ \frac{-2x_2}{(2x_1 + 7x_2)^2} & \frac{2x_1 - 7x_2}{(2x_1 + 7x_2)^2} \end{pmatrix}$$

Now at each steady state

At $(0, \frac{1}{7})$ $J = \begin{pmatrix} \frac{7}{5} & 0 \\ -\frac{2}{7} & 0 \end{pmatrix}$

$$\det(J - \lambda I) = \left(\frac{7}{5} - \lambda \right)(-\lambda) - 0 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = \frac{7}{5} = 1.4$$

$$|\lambda_2| = 1.4 > 1$$

$(0, \frac{1}{7})$ is unstable.

At $(\frac{1}{3}, 0)$ $J = \begin{pmatrix} 0 & -\frac{5}{3} \\ 0 & \frac{3}{2} \end{pmatrix}$

$$\det(J - \lambda I) = \lambda \left(\frac{3}{2} - \lambda \right) - 0 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 1.5$$

$$|\lambda_2| = 1.5 > 1$$

so $(\frac{1}{3}, 0)$ is unstable.

At $(\frac{2}{11}, \frac{1}{11})$ $J = \begin{pmatrix} \frac{5}{11} & -\frac{10}{11} \\ -\frac{14}{11} & \frac{4}{11} \end{pmatrix}$ $\det(J - \lambda I) = \left(\frac{5}{11} - \lambda \right) \left(\frac{4}{11} - \lambda \right) - \frac{20}{121} = 0$

$$\Rightarrow \lambda^2 - \frac{9}{11}\lambda + \frac{20}{121} = 0$$

$$(4\lambda - 5)(5\lambda - 4) = 0$$

$$4\lambda = 5 \quad 5\lambda = 4$$

$$\lambda = \frac{5}{4} \quad \lambda = \frac{4}{5}$$

TPD $\leftarrow$

$$s_0 \quad \cancel{(44 - 121\lambda)} \cancel{(55 - 121\lambda)} - \cancel{70} = 0$$

$$(4 - 11\lambda)(5 - 11\lambda) - 20 = 0$$

$$\Rightarrow 121\lambda^2 - 99\lambda + 20 - 20 = 0$$

$$121\lambda^2 - 99\lambda = 0$$

$$\lambda(121\lambda - 99) = 0$$

$$\Rightarrow \lambda_1 = 0 \quad \lambda_2 = \frac{99}{121} = \frac{9}{11}$$

$$|\lambda_1| < |\lambda_2| < 1$$

So $(\frac{2}{11}, \frac{1}{11})$ is stable.