Homework for Lecture 13 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 20, 2025.

Subject: hw13

with an attachment hw13FirstLast.pdf

1. a Find all the steady-states of the system

$$a_1(n+1) = \frac{a_1(n)}{3 + a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 + a_1(n)}$$

Which of them is a stable steady-state?

2. a Find all the steady-states of the system

$$a_1(n+1) = \frac{a_1(n)}{3a_1(n) + 5a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2a_1(n) + 7a_2(n)}$$

Which of them is a stable steady-state?

Yan Dong.

1. a Find all the steady-states of the system

$(x, y) \to \left(\frac{x}{3+y}, \frac{y}{2+x} \right)$

$$a_1(n+1) = \frac{a_1(n)}{3 + a_2(n)}$$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$a_2(n+1) = \frac{a_2(n)}{2 + a_1(n)}$$

Which of them is a stable steady-state? $(0,0)$

$$J = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{x}{3+y}\right) & \frac{\partial}{\partial y}\left(\frac{x}{3+y}\right) \\ \frac{\partial}{\partial x}\left(\frac{y}{2+x}\right) & \frac{\partial}{\partial y}\left(\frac{y}{2+x}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3+y-0}{(3+y)^2} & \frac{0-x}{(3+y)^2} \\ \frac{0-y}{(2+x)^2} & \frac{2+x-0}{(2+x)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3+y} & -\frac{x}{(3+y)^2} \\ -\frac{y}{(2+x)^2} & \frac{1}{2+x} \end{bmatrix}$$

$\begin{cases} x = \frac{x}{3+y} \\ y = \frac{y}{2+x} \end{cases} \Rightarrow \begin{cases} x\left(1 - \frac{1}{3+y}\right) = 0 \\ \text{or} \\ y\left(1 - \frac{1}{2+x}\right) = 0 \end{cases}$

$x \neq -2, \; y \neq -3.$

$\begin{cases} x \neq 0 \\ 1 - \frac{1}{3+y} = 0 \end{cases}$ or $\begin{cases} y \neq 0 \\ 1 - \frac{1}{2+x} = 0 \end{cases}$ or $\begin{cases} x = 0 \\ y = \frac{y}{2} \end{cases}$ or $\begin{cases} y = 0 \\ x = \frac{x}{3} \end{cases}$

so $(-1, -2)$, $(0, 0)$ are steady state

$$J(0,0) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Eigenvalues are entrices.

so, $\lambda_1 = 1/3$, $\lambda_2 = \frac{1}{2}$.

$|\lambda_1| < 1$ and $|\lambda_2| < 1$, thus $(0, 0)$ is stable steady-state.

$$J(-1,-2) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$\det(J(-1,-2)) = (1-\lambda)(1-\lambda) - 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 1 = 0$

$\Rightarrow \lambda = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

so, $\lambda_1 = 1 + \sqrt{2}$ $\quad \lambda_2 = 1 - \sqrt{2}$

Since $|\lambda_1| = |1+\sqrt{2}| > 1$ $\quad |\lambda_2| = |1 - \sqrt{2}| < 1$

so, $(-1, -2)$ is unstable steady-state

2. a Find all the steady-states of the system

$$a_1(n+1) = \frac{a_1(n)}{3a_1(n) + 5a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2a_1(n) + 7a_2(n)}$$

Which of them is a stable steady-state?

$(x, y) \to \left(\frac{x}{3x+5y}, \frac{y}{2x+7y}\right)$

$\begin{cases} x = \frac{x}{3x+5y} \\ y = \frac{y}{2x+7y} \end{cases} \Rightarrow \begin{cases} x\left(1 - \frac{1}{3x+5y}\right) = 0 \\ y\left(1 - \frac{1}{2x+7y}\right) = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 0 \\ y\left(1 - \frac{1}{7y}\right) = 0 \end{cases}$ or $\begin{cases} y = 0 \\ x\left(1 - \frac{1}{3x}\right) = 0 \end{cases}$ or $\begin{cases} 3x + 5y = 1 \\ 2x + 7y = 1 \end{cases}$

so $\left(0, \frac{1}{7}\right)$, $\left(\frac{1}{3}, 0\right)$, $\left(\frac{2}{11}, \frac{1}{11}\right)$, are steady-state but $(0,0)$ is impossible. b/c $\begin{cases} 3x+5y \neq 0 \\ 2x+7y \neq 0 \end{cases}$

$$J = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{x}{3x+5y}\right) & \frac{\partial}{\partial y}\left(\frac{x}{3x+5y}\right) \\ \frac{\partial}{\partial x}\left(\frac{y}{2x+7y}\right) & \frac{\partial}{\partial y}\left(\frac{y}{2x+7y}\right) \end{bmatrix} = \begin{bmatrix} \frac{5y}{(3x+5y)^2} & -\frac{5x}{(3x+5y)^2} \\ \frac{-2y}{(2x+7y)^2} & \frac{2x}{(2x+7y)^2} \end{bmatrix}$$

$$J\left(0, \frac{1}{7}\right) = \begin{bmatrix} \frac{5(1/7)}{25/49} & 0 \\ \frac{-2(1/7)}{1} & 0 \end{bmatrix} = \begin{bmatrix} 7/5 & 0 \\ -2/7 & 0 \end{bmatrix}$$

char. $(7/5 - \lambda)(-\lambda) - 0 = 0$

$\Rightarrow \lambda_1 = 7/5,\ \lambda_2 = 0$ $\quad |\lambda_1| = 7/5 > 1$ $\quad$ so, $\left(0, \frac{1}{7}\right)$ is unstable steady-state

$$J\left(\frac{1}{3}, 0\right) = \begin{bmatrix} 0 & -\frac{5/3}{1} \\ 0 & \frac{2/3}{(2/3)^2} \end{bmatrix} = \begin{bmatrix} 0 & -5/3 \\ 0 & 3/2 \end{bmatrix}$$

char. $(0 - \lambda)(3/2 - \lambda) - 0 = 0 \quad \Rightarrow \quad \lambda_1 = 0 \quad \lambda_2 = 3/2$

since $|\lambda_2| = 3/2 > 1$, so, $\left(\frac{1}{3}, 0\right)$ is unstable steady-state.

$$J\left(2/11, 1/11\right) = \begin{bmatrix} 5/11 & -10/11 \\ -2/11 & 4/11 \end{bmatrix}$$

char. $(5/11 - \lambda)(4/11 - \lambda) - \frac{20}{121} = 0$

$\lambda^2 - \frac{9}{11}\lambda + \frac{20}{121} - \frac{20}{121} = 0$

$\lambda\left(\lambda - \frac{9}{11}\right) = 0$

$\lambda_1 = 0,\ \lambda_2 = \frac{9}{11}$

$|\lambda_1| < 1$. $|\lambda_2| = \frac{9}{11} < 1$ $\quad$ both $|\lambda| < 1$,

so, $\boxed{\left(\frac{2}{11}, \frac{1}{11}\right) \text{ is stable steady-state.}}$