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Homework #13

$$1.) \quad a_1(n+1) = \frac{a_1(n)}{f_1 // 3 + a_2(n)}, \quad a_2(n+1) = \frac{a_2(n)}{f_2 // 2 + a_1(n)}$$

$$(x, y) \rightarrow \left(\frac{x}{3+y}, \frac{y}{2+x} \right)$$

$$x = \frac{x}{3+y} \Rightarrow x \left(1 - \frac{1}{3+y} \right) = 0 \quad (0, -2)$$

$$y = \frac{y}{2+x} \Rightarrow y \left(1 - \frac{1}{2+x} \right) = 0 \quad (-2, 0)$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{3+y} & -\frac{x}{(3+y)^2} \\ -\frac{y}{(2+x)^2} & \frac{1}{2+x} \end{bmatrix}$$

$$\frac{\partial}{\partial x} \left(\frac{y}{2+x} \right) = y(2+x)^{-1} = -y(2+x)^{-2}$$

$$\frac{\partial}{\partial y} \left(\frac{x}{3+y} \right) = x(3+y)^{-1} = -x(3+y)^{-2}$$

$$J(0, -2) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad |J - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix}$$

$$(1-\lambda)\left(\frac{1}{2}-\lambda\right) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - \frac{1}{2})(\lambda - 1) = 0$$

$$\lambda = \frac{1}{2}, 1 \Rightarrow (0, -2) \text{ is semi-stable}$$

$$J(-2, 0) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \quad |J - \lambda I| = \begin{vmatrix} \frac{1}{3}-\lambda & \frac{2}{3} \\ 0 & -\lambda \end{vmatrix}$$

$$(\frac{1}{3}-\lambda)\lambda = \lambda^2 - \frac{1}{3}\lambda = 0 \Rightarrow \lambda = 0, \frac{1}{3}$$

$$\Rightarrow (-2, 0) \text{ is a SSS}$$

$$\begin{aligned}
 2.) \quad & \begin{cases} a_1(n+1) = a_1(n) / 3a_1(n) + 5a_2(n) \\ a_2(n+1) = a_2(n) / 2a_1(n) + 7a_2(n) \end{cases} \\
 & \begin{cases} f_1 = x / 3x + 5y \\ f_2 = y / 2x + 7y \end{cases} \quad (x, y) \rightarrow (x / 3x + 5y, y / 2x + 7y)
 \end{aligned}$$

Steady States:

$$x = x / 3x + 5y, \quad y = y / 2x + 7y$$

$$x(1 - 1/3x + 5y) = 0, \quad y(1 - 1/2x + 7y) = 0$$

$$\text{if } x = 0, 1 = 1/7y \Rightarrow y = 1/7 \quad (0, 1/7)$$

$$\text{if } y = 0, 1 = 1/3x \Rightarrow x = 1/3 \quad (1/3, 0)$$

$$1 = 1/3x + 5y, \quad 1 = 1/2x + 7y \quad (2/11, 1/11)$$

$$3/2 = 3x + 2y \Rightarrow 1/2 = 1/2y \Rightarrow y = 1/11$$

$$1 = 3x + 5/11 \Rightarrow 6/11 = 3x \Rightarrow x = 2/11$$

$$J = \begin{bmatrix} \partial f_1 / \partial x & \partial f_1 / \partial y \\ \partial f_2 / \partial x & \partial f_2 / \partial y \end{bmatrix} = \begin{bmatrix} 5y / (3x + 5y)^2 & -2y / (2x + 7y)^2 \\ -5x / (3x + 5y)^2 & 2x / (2x + 7y)^2 \end{bmatrix}$$

$$\partial / \partial x f_1 = \frac{3x + 5y - x(3)}{(3x + 5y)^2} \quad \partial / \partial x f_2 = \frac{2y}{(2x + 7y)^2}$$

$$\partial / \partial y f_1 = \frac{-5x}{(3x + 5y)^2} \quad \partial / \partial y f_2 = \frac{2x + 7y - 7y}{(2x + 7y)^2}$$

$$\partial / \partial y f_2 = \frac{2x + 7y - 7y}{(2x + 7y)^2}$$

$$J(0, 1/7) = \begin{bmatrix} (5/7) / (5/7)^2 & (-4/7) / 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7/5 & -4/7 \\ 0 & 0 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} 7/5 - \lambda & -4/7 \\ 0 & -\lambda \end{vmatrix} = -\lambda(7/5 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, 7/5$$

$\Rightarrow (0, 1/7)$ is unstable

$$J = \begin{bmatrix} 5y/(3x+5y)^2 & -2y/(2x+7y)^2 \\ 5x/(3x+5y)^2 & 2x/(2x+7y)^2 \end{bmatrix}$$

$$J(1/3, 0) = \begin{bmatrix} 0 & 0 \\ -5/3 & 2/3(1/3)^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -5/3 & 3/2 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ -5/3 & 3/2 - \lambda \end{vmatrix} = -\lambda(3/2 - \lambda) = 0$$

$\Rightarrow (1/3, 0)$ is unstable

$$J(2/11, 1/11) = \begin{bmatrix} (5/11)/(6/11+5/11)^2 & (2/11)/(4/11+7/11)^2 \\ (-10/11)/(6/11+5/11)^2 & (4/11)/(4/11+7/11)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/11 & -2/11 \\ -10/11 & 4/11 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} 5/11 - \lambda & -2/11 \\ -10/11 & 4/11 - \lambda \end{vmatrix} = (-\lambda + 5/11)(-\lambda + 4/11) - 20/121$$

$$= \lambda^2 - 9/11\lambda + 20/121 - 20/121$$

$$\lambda^2 - 9/11\lambda = 0 \Rightarrow \lambda = 0, 9/11$$