HW 12 Dyn Mod in Bio

1. a Find all the steady-states of the system

$$a_1(n+1) = \frac{a_1(n)}{3 + a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 + a_1(n)}$$

Which of them is a stable steady-state?

$$(X_1, X_2) \to \left(\frac{X_1}{3+X_2}, \frac{X_2}{2+X_1}\right)$$

X_1, X_2

$(-1, -2)$

$$X_1 = \frac{X_1}{3+X_2} \qquad X_2 = \frac{X_2}{2+X_1}$$

fixed pts

$(0,0)$

$(0,-2)$

$(-1,0)$

$(-1,-2)$

$$0 = \frac{X_1}{3+X_2} - X_1 \qquad 0 = \frac{X_2}{2+X_1} - X_2$$

$$0 = X_1\left(\frac{1}{3+X_2} - 1\right) \qquad 0 = X_2\left(\frac{1}{2+X_1} - 1\right)$$

$X_2 = 0$ $\quad 1 = \frac{1}{2+X_1}$

$X_1 = 0$ $\quad 1 = \frac{1}{3+X_2}$ $\quad 2 + X_1 = 1$

$(-1, 0)$ $\quad X_1 = -1$

$3 + X_2 = 1$

$(0, -2)$ $\quad X_2 = -2$

$\frac{1}{3+X_2} = 1 \qquad 1 = \frac{1}{2+X_1}$

$1 = 3 + X_2 \qquad 2 + X_1 = 1$

$X_2 = -2 \qquad X_1 = -1$

$(-1, -2)$

$$J = \begin{bmatrix} \frac{1}{3+X_2} & \frac{-X_1}{(3+X_2)^2} \\ \frac{-X_2}{(2+X_1)^2} & \frac{1}{2+X_1} \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\left(\tfrac{1}{3} - \lambda\right)\left(\tfrac{1}{2} - \lambda\right) - 0 = 0$$

$$\frac{1}{6} - \frac{1}{3}\lambda - \frac{1}{2}\lambda + \lambda^2 = 0$$

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0$$

$$\left(\lambda - \tfrac{2}{6}\right)\left(\lambda - \tfrac{3}{6}\right) = 0$$

$\lambda = \frac{2}{6}, \frac{3}{6}$

$\hookrightarrow \lambda = \frac{1}{3}, \frac{1}{2}$

both are steady SS b/c <1

$$J(-1,0) = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ 0 & 1 \end{bmatrix}$$

$$\left(\tfrac{1}{3}-\lambda\right)(1-\lambda)=0$$
$$\tfrac{1}{3}-\tfrac{1}{3}\lambda-\lambda+\lambda^2=0$$
$$\lambda^2-\tfrac{4}{3}\lambda+\tfrac{1}{3}=0$$
$$(\lambda-1)\left(\lambda-\tfrac{1}{3}\right)=0$$
$$\lambda=1,\tfrac{1}{3}$$

$$J(0,-2) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(1-\lambda)\left(\tfrac{1}{2}-\lambda\right)-0=0$$
$$\tfrac{1}{2}-\lambda-\tfrac{1}{2}\lambda+\lambda^2=0$$
$$\lambda^2-\tfrac{3}{2}\lambda+\tfrac{1}{2}=0$$
$$(\lambda-1)\left(\lambda-\tfrac{1}{2}\right)=0$$
$$\lambda=1,\tfrac{1}{2}$$

$$J(-1,-2) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$(1-\lambda)(1-\lambda)-2=0$$
$$1-2\lambda+\lambda^2-2=0$$
$$\lambda^2-2\lambda-1=0$$
$$x=1\pm\sqrt{2} \qquad \sqrt{2}\approx 1.41$$
$$1-1.41=-0.41$$
$$1+1.41=2.41$$

The only steady stable state is at (0,0) b/c both $\lambda<1$

2. a Find all the steady-states of the system

$$a_1(n+1) = \frac{a_1(n)}{3a_1(n) + 5a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2a_1(n) + 7a_2(n)}$$

Which of them is a stable steady-state?

$$(X_1, X_2) \to \left(\frac{X_1}{3X_1 + 5X_2}, \frac{X_2}{2X_1 + 7X_2} \right)$$

$2X_1 + 7X_2 - (X_2)(7)$

$(0,0)$

$$X_1 = \frac{X_1}{3X_1 + 5X_2} \qquad X_2 = \frac{X_2}{2X_1 + 7X_2}$$

$X_1 = 0 \quad X_2 = 1/7$

$X_1 = 1/3 \quad X_2 = 0$

$$X_1\left(1 - \frac{1}{3X_1 + 5X_2}\right) = 0 \qquad X_2\left(1 - \frac{1}{2X_1 + 7X_2}\right) = 0$$

$$1 = \frac{1}{3X_1 + 5X_2} \qquad 1 = \frac{1}{2X_1 + 7X_2}$$

$$3X_1 + 5X_2 = 1 \qquad 2X_1 + 7X_2 = 1$$

$$X_1 = \frac{1}{3} - \frac{5X_2}{3} \to 3\left(2\left(\frac{1}{3} - \frac{5X_2}{3}\right) + 7X_2 = 1\right)$$

Fixed pts

$(1/3, 0)$

$(0, 1/7)$

$(2/11, 1/11)$

$\cancel{(0,0)}$ divide by 0

$$3X_1 + 5\left(\frac{1}{11}\right) = 1$$

$$2 - 10X_2 + 21X_2 = 3$$

$$11X_2 = 1$$

$$X_2 = \frac{1}{11}$$

$$3X_1 = 1 - \frac{5}{11}$$

$$3X_1 = \frac{6}{11}$$

$$\left(\frac{2}{11}, \frac{1}{11}\right)$$

$$X_1 = \frac{2}{11}$$

$$J = \begin{bmatrix} \frac{5X_2}{(3X_1+5X_2)^2} & \frac{-5X_1}{(3X_1+5X_2)^2} \\ \frac{-2X_2}{(2X_1+7X_2)^2} & \frac{2X_1}{(2X_1+7X_2)^2} \end{bmatrix}$$

$$J(1/3, 0) = \begin{bmatrix} 0 & -5/3 \\ 0 & 3/2 \end{bmatrix}$$

$$(-\lambda)(3/2 - \lambda) = 0$$

$$\lambda = 0, \ \lambda = 3/2$$

$$J(0, 1/7) = \begin{bmatrix} 5/7 & 0 \\ -2/7 & 0 \end{bmatrix}$$

$$\left(\frac{5}{7} - \lambda\right)(-\lambda) = 0$$

$$\lambda = 0, \ \lambda = \frac{5}{7}$$

$$J(2/11, 1/11) = \begin{bmatrix} 5/11 & -\frac{10}{11} \\ -\frac{2}{11} & 4/11 \end{bmatrix}$$

$$\left(\frac{5}{11} - \lambda\right)\left(\frac{4}{11} - \lambda\right) - \left(\frac{-10}{11}\right)\left(\frac{-2}{11}\right) = 0$$

$$\cancel{\frac{20}{121}} - \frac{5}{11}\lambda - \frac{4}{11}\lambda + \lambda^2 - \cancel{\frac{20}{121}} = 0$$

$$\lambda^2 - \frac{9}{11}\lambda = 0$$

$$\lambda\left(\lambda - \frac{9}{11}\right) = 0$$

$$\lambda = 0, \ \lambda = \frac{9}{11} \Big\} \ \boxed{\text{only steady ss is } (2/11, 1/11) \text{ as both } \lambda < 1}$$