

Homework for Lecture 12 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as .pdf file) to

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by 8:00pm Monday, Oct. 20, 2025.

Subject: hw12

with an attachment hw8FirstLast.pdf and/or hw12FirstLast.txt

Using

<http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB12.txt>

1. What is the ultimate period of

`evalf(Orb((1-x)*(2-x)*(3-x)/12,x,0.5,1000,1010),10);?`

By Playing 'high-low', estimate the cut-offs r_1, r_2, r_3, r_4 such that

the recurrence

$$a(n+1) = (1 - a(n))(2 - a(n))(3 - a(n))/r \quad ,$$

has

- ultimate period 1 for $r_1 < r$
- ultimate period 2 for $r_2 < r < r_1$
- ultimate period 4 for $r_3 < r < r_2$

(challenging!):

- ultimate period 8 for $r_4 < r < r_3$

2.

In the Maple package <http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB12.txt>

Run the following line 20 times

```
T:=RT([x,y],30); SSg(T,[x,y]); ORB(T,[x,y,z],[100., 200.],1000,1020);
```

Here the first command, `T:=RT([x,y],30);` generates a random rational transformation in R^2 , the second command, `SSg(T,[x,y]);`, finds steady-states (i.e. fixed points of the transformation T , using

Maple's `solve` command, but unfortunately not all, and the last command, `ORB(T, [x,y], [100., 200.], 1000, 1020)`; finds the long-term behavior.

If all goes well, the ultimate behavior of the orbit should be ONE of the outputs of `SSg`, but sometimes it misses it. Out of the 20 tries, list all the cases where

```
ORB(T, [x,y], [100., 200.], 1000, 1020)[-1];
```

was not part of the output of

```
SSg(T, [x,y]);
```

3.

In the Maple package <http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB12.txt>

Run the following line 20 times

```
T:=RT([x,y,z], 10); SSg(T, [x,y,z]); ORB(T, [x,y,z], [100., 200., 300.], 1000, 1020);
```

Here the first command, `T:=RT([x,y,z], 10)`; generates a random rational transformation in R^3 , the second command, `SSg(T, [x,y,z])`; finds steady-states (i.e. fixed points of the transformation `T`, using Maple's `solve` command, but unfortunately not all, and the last command, `ORB(T, [x,y,z], [100., 200., 300.], 1000, 1020)`; finds the long-term behavior.

If all goes well, the ultimate behavior of the orbit should be ONE of the outputs of `SSg`, but sometimes it misses it. Out of the 20 tries, list all the cases where

```
ORB(T, [x,y,z], [100., 200., 300.], 1000, 1020)[-1];
```

was not part of the output of

```
SSg(T, [x,y,z]);
```

Problem 1

```
> read("DMB12.txt");
```

For a list of the Main procedures type: Help12(); for help with a specific procedure type: Help12 (ProcedureName); for example Help12(Feig); (1)

```
> f := (x, r) →  $\frac{((1-x) \cdot (2-x) \cdot (3-x))}{r}$  :
```

```
orbit := proc(x0, r, n)
```

```
    local x, i;
```

```
    x := x0;
```

```
    for i to n do
```

```
        x := f(x, r);
```

```
    end do;
```

```
    return x;
```

```
end proc;
```

```
> f := (x, r) → (1 - x) * (2 - x) * (3 - x) / r :
```

```
rvalues := [12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1] :
```

```
x0 := 0.5 :
```

```
for r in rvalues do
```

```
    printf("r = %.3f\n", r);
```

```
    x := x0;
```

```
    for i from 1 to 1000 do
```

```
        x := f(x, r);
```

```
    end do;
```

```
    for i from 1001 to 1010 do
```

```
        x := f(x, r);
```

```
        printf("x[%d] = %.10f\n", i, x);
```

```
    end do;
```

```
end do;
```

```
r = 12.000
```

```
x[1001] = 0.2804250992
```

```
x[1002] = 0.2804250992
```

```
x[1003] = 0.2804250992
```

```
x[1004] = 0.2804250992
```

```
x[1005] = 0.2804250992
```

```
x[1006] = 0.2804250992
```

```
x[1007] = 0.2804250992
```

```
x[1008] = 0.2804250992
```

```
x[1009] = 0.2804250992
```

```
x[1010] = 0.2804250992
```

← converges to a single pt
(1 period cycle)

```
r = 11.000
x[1001] = 0.2953460398
x[1002] = 0.2953460398
x[1003] = 0.2953460398
x[1004] = 0.2953460398
x[1005] = 0.2953460398
x[1006] = 0.2953460398
x[1007] = 0.2953460398
x[1008] = 0.2953460398
x[1009] = 0.2953460398
x[1010] = 0.2953460398 ← 11
r = 10.000
x[1001] = 0.3120964620
x[1002] = 0.3120964620
x[1003] = 0.3120964620
x[1004] = 0.3120964620
x[1005] = 0.3120964620
x[1006] = 0.3120964620
x[1007] = 0.3120964620
x[1008] = 0.3120964620
x[1009] = 0.3120964620
x[1010] = 0.3120964620 ← 11
r = 9.000
x[1001] = 0.3310673322
x[1002] = 0.3310673322
x[1003] = 0.3310673322
x[1004] = 0.3310673322
x[1005] = 0.3310673322
x[1006] = 0.3310673322
x[1007] = 0.3310673322
x[1008] = 0.3310673322
x[1009] = 0.3310673322
x[1010] = 0.3310673322 ← 11
r = 8.000
x[1001] = 0.3527798404
x[1002] = 0.3527798404
x[1003] = 0.3527798404
x[1004] = 0.3527798404
x[1005] = 0.3527798404
x[1006] = 0.3527798404
x[1007] = 0.3527798404
x[1008] = 0.3527798404
```


x[1009] = 0.3527798404

x[1010] = 0.3527798404 ← " "

r = 7.000

x[1001] = 0.3779491508

x[1002] = 0.3779491923

x[1003] = 0.3779491515

x[1004] = 0.3779491917

x[1005] = 0.3779491521

x[1006] = 0.3779491911

x[1007] = 0.3779491527

x[1008] = 0.3779491905

x[1009] = 0.3779491532

x[1010] = 0.3779491899

r = 6.000

x[1001] = 0.0000000000

x[1002] = 1.0000000000

x[1003] = 0.0000000000

x[1004] = 1.0000000000

x[1005] = 0.0000000000

x[1006] = 1.0000000000

x[1007] = 0.0000000000

x[1008] = 1.0000000000

x[1009] = 0.0000000000

x[1010] = 1.0000000000

r = 5.000

x[1001] = -0.0761579194

x[1002] = 1.3745958007

x[1003] = -0.0761579194

x[1004] = 1.3745958007

x[1005] = -0.0761579194

x[1006] = 1.3745958007

x[1007] = -0.0761579194

x[1008] = 1.3745958007

x[1009] = -0.0761579194

x[1010] = 1.3745958007

r = 4.000

x[1001] = -0.0688699477

x[1002] = 1.6965886244

x[1003] = -0.0688699477

x[1004] = 1.6965886244

x[1005] = -0.0688699477

x[1006] = 1.6965886244

} tiny alternating/ messed up
but, not a clear 2 cycle

} 2 cycle period
0 and 1 → alternating btwn

} " "

} " "

```

x[1007] = -0.0688699477
x[1008] = 1.6965886244
x[1009] = -0.0688699477
x[1010] = 1.6965886244

```

```
r = 3.000
```

```

x[1001] = 1.5933566277
x[1002] = -0.1131337664
x[1003] = 2.4409049840
x[1004] = 0.1183980958

```

```

x[1005] = 1.5933566277
x[1006] = -0.1131337664
x[1007] = 2.4409049840
x[1008] = 0.1183980958
x[1009] = 1.5933566277
x[1010] = -0.1131337664

```

```
r = 2.000
```

```

x[1001] = 2.0616532130
x[1002] = 0.0307094309
x[1003] = 2.8339128568
x[1004] = 0.1270004864
x[1005] = 2.3488604922
x[1006] = 0.1532014497
x[1007] = 2.2260062078
x[1008] = 0.1072310403
x[1009] = 2.4441082687
x[1010] = 0.1782579190

```

```
r = 1.000
```

```

x[1001] = Inf
x[1002] = -Inf
x[1003] = Inf
x[1004] = -Inf
x[1005] = Inf
x[1006] = -Inf
x[1007] = Inf
x[1008] = -Inf
x[1009] = Inf
x[1010] = -Inf

```

4 cycle period → alt.
btwn 4 numbers

8 cycle must be in
btwn 3 and 2

complete chaos

test $r=2.5$

```

r = 3.000
x[1001] = 1.5933566277
x[1002] = -0.1131337664
x[1003] = 2.4409049840
x[1004] = 0.1183980958
x[1005] = 1.5933566277
x[1006] = -0.1131337664
x[1007] = 2.4409049840
x[1008] = 0.1183980958
x[1009] = 1.5933566277
x[1010] = -0.1131337664
r = 2.500
x[1001] = 1.9079349814
x[1002] = -0.0365138714
x[1003] = 2.5638803379
x[1004] = 0.1538353447
x[1005] = 1.7784650105
x[1006] = -0.0842650201
x[1007] = 2.7880468459
x[1008] = 0.1194622811
x[1009] = 1.9079349814
x[1010] = -0.0365138714
r = 2.000
x[1001] = 2.0616532130
x[1002] = 0.0307094309
x[1003] = 2.8339128568
x[1004] = 0.1270004864
x[1005] = 2.3488604922
x[1006] = 0.1532014497
x[1007] = 2.2260062078
x[1008] = 0.1072310403
x[1009] = 2.4441082687
x[1010] = 0.1782579190

```

8 period
cycle

$6.7 < r \leftarrow 1 \text{ cycle}$

$3.2 < r < 6.7 \leftarrow 2 \text{ cycle}$

$2.5 < r < 3.2 \leftarrow 4 \text{ cycle}$

$2.4 < r < 2.5 \leftarrow 8 \text{ cycle}$

Formula Assistant	
Palettes	
Workbooks	
Execution	
Variables	
Variable	Value
ChaosStory	proc() local L...
Digits	100
f	(x, r) -> (1-x)*L...
Feig	proc() [3, 3.4...
Help12	proc() if_npa...
Help12a	proc() if_npa...
i	1011
L	[(1, 2083333...
Orb	proc(f, x, x0, ...
ORB	proc(F, x, x0, ...
orbit	proc(x0, r, n) L...
Orbk	proc(k, z, f, L...
r	3.1
RecToT	proc(k, z, f, L...
RT	proc(x, k) loc...
rvalues	[4, 3.9, 3.8, 3...
SS	proc(f, x) [sol...
SSg	proc(T, x) loca...
SSS	proc(f, x) loca...
x	-106119595...
x0	.5

r = 3.400
x[1001] = 1.8796896101
x[1002] = -0.0348732195
x[1003] = 1.8796896101
x[1004] = -0.0348732195
x[1005] = 1.8796896101
x[1006] = -0.0348732195
x[1007] = 1.8796896101
x[1008] = -0.0348732195
x[1009] = 1.8796896101
x[1010] = -0.0348732195
r = 3.300
x[1001] = 2.0231095350
x[1002] = 0.0069991495
x[1003] = 1.7949402852
x[1004] = -0.0595263815
x[1005] = 2.0231095341
x[1006] = 0.0069991492
x[1007] = 1.7949402861
x[1008] = -0.0595263813
x[1009] = 2.0231095333
x[1010] = 0.0069991490
r = 3.200
x[1001] = 1.6677467567
x[1002] = -0.0923672097
x[1003] = 2.2087554889
x[1004] = 0.0623931762
x[1005] = 1.6677467567
x[1006] = -0.0923672097
x[1007] = 2.2087554889
x[1008] = 0.0623931762
x[1009] = 1.6677467567
x[1010] = -0.0923672097
r = 3.100
x[1001] = 1.6131246294
x[1002] = -0.1061195956
x[1003] = 2.3342189727
x[1004] = 0.0957696572
x[1005] = 1.6131246294
x[1006] = -0.1061195956
x[1007] = 2.3342189727
x[1008] = 0.0957696572
x[1009] = 1.6131246294
x[1010] = -0.1061195956

after
2.4
complete
chaos

r = 7.000
x[1001] = 0.3779491508
x[1002] = 0.3779491923
x[1003] = 0.3779491515
x[1004] = 0.3779491917
x[1005] = 0.3779491521
x[1006] = 0.3779491911
x[1007] = 0.3779491527
x[1008] = 0.3779491905
x[1009] = 0.3779491532
x[1010] = 0.3779491899
r = 6.900
x[1001] = 0.3800685749
x[1002] = 0.3813125643
x[1003] = 0.3800746165
x[1004] = 0.3813065468
x[1005] = 0.3800805995
x[1006] = 0.3813005877
x[1007] = 0.3800865245
x[1008] = 0.3812946864
x[1009] = 0.3800923922
x[1010] = 0.3812888423
r = 6.800
x[1001] = 0.2620629152
x[1002] = 0.5163782463
x[1003] = 0.2620629120
x[1004] = 0.5163782500
x[1005] = 0.2620629089
x[1006] = 0.5163782537
x[1007] = 0.2620629059
x[1008] = 0.5163782572
x[1009] = 0.2620629030
x[1010] = 0.5163782607
r = 6.700
x[1001] = 0.1883615430
x[1002] = 0.6170478166
x[1003] = 0.1883615430
x[1004] = 0.6170478166
x[1005] = 0.1883615430
x[1006] = 0.6170478166
x[1007] = 0.1883615430
x[1008] = 0.6170478166
x[1009] = 0.1883615430
x[1010] = 0.6170478166

Clipboard
Find/Replace
Worksheet
Selection
Code
Key
Execution
Workbooks
x[1005] = 2.6894380899
x[1006] = 0.1391273126
x[1007] = 1.7627096197
x[1008] = -0.0861266678
x[1009] = 2.6894380899
x[1010] = 0.1391273126
r = 2.500
x[1001] = 1.9079349814
x[1002] = -0.0365138714
x[1003] = 2.5638803379
x[1004] = 0.1538353447
x[1005] = 1.7784650105
x[1006] = -0.0842650201
x[1007] = 2.7880468459
x[1008] = 0.1194622811
x[1009] = 1.9079349814
x[1010] = -0.0365138714
r = 2.400
x[1001] = Inf
x[1002] = -Inf
x[1003] = Inf
x[1004] = -Inf
x[1005] = Inf
x[1006] = -Inf
x[1007] = Inf
x[1008] = -Inf
x[1009] = Inf
x[1010] = -Inf
r = 2.300
x[1001] = Inf
x[1002] = -Inf
x[1003] = Inf
x[1004] = -Inf
x[1005] = Inf
x[1006] = -Inf
x[1007] = Inf
x[1008] = -Inf
x[1009] = Inf
x[1010] = -Inf
r = 2.200
x[1001] = 1.9645083611
x[1002] = -0.0161122417
x[1003] = 2.8085438487
x[1004] = 0.1272562918
x[1005] = 2.1342204382
x[1006] = 0.0594102000

Problem 2

```
> read("DMB12.txt");
For a list of the Main procedures type: Help12(); for help with a specific procedure type: Help12
(ProcedureName); for example Help12(Feig);
```

```
> T := RT([x, y], 30);
T := [ (12 + 22 x + 5 y) / (4 + 11 x + 3 y), (24 + 22 x + 30 y) / (3 + 25 x + 23 y) ]
```

```
> SS := SSg(T, [x, y]);
SS := [[2.088282786, 1.279928303]]
```

```
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [2.088282786, 1.279928313]
```

```
> evalf(SS);
[[2.088282786, 1.279928303]]
```

```
> evalf(o);
[2.088282786, 1.279928313]
```

```
> T := RT([x, y], 30);
T := [ (26 + 23 x + 3 y) / (1 + 11 x + 11 y), (16 + 11 x + 26 y) / (19 + 24 x + 9 y) ]
```

```
> SS := SSg(T, [x, y]);
SS := [[2.347467067, 0.7461026524]]
```

```
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [2.347467067, 0.7461026535]
```

```
> evalf(SS);
[[2.347467067, 0.7461026524]]
```

```
> evalf(o);
[2.347467067, 0.7461026535]
```

```
> T := RT([x, y], 30);
T := [ (20 + 10 x + 15 y) / (13 + 22 x + 2 y), (3 + 15 x + 24 y) / (9 + 26 x + 14 y) ]
```

```
> SS := SSg(T, [x, y]);
SS := [[0.5408318860, -0.8580424636]]
```

```
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [1.103609195, 0.7899610497]
```

```
> evalf(SS);
[[0.5408318860, -0.8580424636]]
```

```
> evalf(o);
[1.103609195, 0.7899610497]
```

```

>
> T := RT([x, y], 30);
T := [ (8 + 29 x + 22 y) / (13 + 25 x + 12 y), (15 + 10 x + 21 y) / (14 + 7 x + 23 y) ] (17)
> SS := SSg(T, [x, y]);
SS := [ [-0.937055694, 0.8706576678] ] (18)
> o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1];
o := [1.185796056, 1.052715492] (19)
> evalf(SS);
[ [-0.937055694, 0.8706576678] ] (20)
> evalf(o);
[1.185796056, 1.052715492] (21)
>
> T := RT([x, y], 30);
T := [ (28 + 5 x + 14 y) / (24 + 18 x + 12 y), (7 + 30 x + 26 y) / (4 + 17 x + 29 y) ] (22)
> SS := SSg(T, [x, y]);
SS := [ [0.9026628745, 1.204843624] ] (23)
> o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1];
o := [0.9026628752, 1.204843624] (24)
> evalf(SS);
[ [0.9026628745, 1.204843624] ] (25)
> evalf(o);
[0.9026628752, 1.204843624] (26)
>
>
> T := RT([x, y], 30);
T := [ (22 + 12 x + 15 y) / (22 + 3 x + 25 y), (23 + 3 x + 27 y) / (11 + 3 x + 24 y) ] (27)
> SS := SSg(T, [x, y]);
SS := [ [0.909338337, 1.348158543] ] (28)
> o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1];
o := [0.9093383361, 1.348158543] (29)
> evalf(SS);
[ [0.909338337, 1.348158543] ] (30)
> evalf(o);
[0.9093383361, 1.348158543] (31)
>

```


$$\begin{aligned}
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{21 + 10x + 23y}{24 + 5x + 21y}, \frac{7 + 22x + 8y}{27 + 26x + 21y} \right] \quad (32) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[1.086047915, 0.5294015064]] \quad (33) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [1.086047916, 0.5294015064] \quad (34) \\
& \text{> } evalf(SS); \\
& \quad [[1.086047915, 0.5294015064]] \quad (35) \\
& \text{> } evalf(o); \\
& \quad [1.086047916, 0.5294015064] \quad (36) \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{11 + 11x + 18y}{12 + 7x + 22y}, \frac{3 + 7x + 24y}{23 + 17x + 16y} \right] \quad (37) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[-2.42266936, 0.3879819419]] \quad (38) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [1.050603808, 0.4347534304] \quad (39) \\
& \text{> } evalf(SS); \\
& \quad [[-2.42266936, 0.3879819419]] \quad (40) \\
& \text{> } evalf(o); \\
& \quad [1.050603808, 0.4347534304] \quad (41) \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{29 + 2x + 20y}{23 + 27x + 20y}, \frac{24 + 9x + 29y}{8 + 19x + 18y} \right] \quad (42) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[0.809087586, 1.483835682]] \quad (43) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [0.8090875886, 1.483835682] \quad (44) \\
& \text{> } evalf(SS); \\
& \quad [[0.809087586, 1.483835682]] \quad (45) \\
& \text{> } evalf(o); \\
& \quad [0.8090875886, 1.483835682] \quad (46) \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{7 + 19x + 29y}{27 + 9x + 15y}, \frac{1 + 11x + 18y}{14 + 4x + 15y} \right] \quad (47)
\end{aligned}$$

$$\begin{aligned}
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[1.069048679, 0.9131385953]] \quad (48) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [1.069048679, 0.9131385953] \quad (49) \\
& \text{> } evalf(SS); \\
& \quad [[1.069048679, 0.9131385953]] \quad (50) \\
& \text{> } evalf(o); \\
& \quad [1.069048679, 0.9131385953] \quad (51) \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{27 + x + 9y}{16 + 16x + 27y}, \frac{16 + 23x + 21y}{16 + 18x + 13y} \right] \quad (52) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[0.6403715550, 1.306776021]] \quad (53) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [0.6403715551, 1.306776021] \quad (54) \\
& \text{> } evalf(SS); \\
& \quad [[0.6403715550, 1.306776021]] \quad (55) \\
& \text{> } evalf(o); \\
& \quad [0.6403715551, 1.306776021] \quad (56) \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{24 + 13x + 16y}{18 + 9x + 9y}, \frac{16 + 5x + 5y}{13 + 5x + 20y} \right] \quad (57) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[-2.531073363, 0.5415628792]] \quad (58) \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1]; \\
& \quad o := [1.452017555, 0.7624157717] \quad (59) \\
& \text{> } evalf(SS); \\
& \quad [[-2.531073363, 0.5415628792]] \quad (60) \\
& \text{> } evalf(o); \\
& \quad [1.452017555, 0.7624157717] \quad (61) \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{26 + 23x + 3y}{1 + 11x + 11y}, \frac{16 + 11x + 26y}{19 + 24x + 9y} \right] \quad (62) \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[2.347467067, 0.7461026524]] \quad (63)
\end{aligned}$$

$$\begin{aligned}
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1]; \\
& \quad o := [2.347467067, 0.7461026535] \tag{64} \\
& \text{> } evalf(SS); \\
& \quad [[2.347467067, 0.7461026524]] \tag{65} \\
& \text{> } evalf(o); \\
& \quad [2.347467067, 0.7461026535] \tag{66} \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{22 + 17x + 6y}{29 + 26x + 14y}, \frac{3 + 30x + 28y}{11 + 6x + 11y} \right] \tag{67} \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[0.6047688614, 2.121562406]] \tag{68} \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1]; \\
& \quad o := [0.6047688614, 2.121562406] \tag{69} \\
& \text{> } evalf(SS); \\
& \quad [[0.6047688614, 2.121562406]] \tag{70} \\
& \text{> } evalf(o); \\
& \quad [0.6047688614, 2.121562406] \tag{71} \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{1 + 21x + 2y}{2 + 17x + 26y}, \frac{20 + x + 25y}{15 + 30x + 15y} \right] \tag{72} \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[0.01837264827, -0.8824358281]] \tag{73} \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1]; \\
& \quad o := [0.1975494793, 1.304090558] \tag{74} \\
& \text{> } evalf(SS); \\
& \quad [[0.01837264827, -0.8824358281]] \tag{75} \\
& \text{> } evalf(o); \\
& \quad [0.1975494793, 1.304090558] \tag{76} \\
& \text{> } \\
& \text{> } T := RT([x, y], 30); \\
& \quad T := \left[\frac{1 + 23x + 15y}{27 + 26x + 19y}, \frac{7 + 9x + 2y}{25 + 19x + 6y} \right] \tag{77} \\
& \text{> } SS := SSg(T, [x, y]); \\
& \quad SS := [[-0.524740111, 0.1626025726]] \tag{78} \\
& \text{> } o := ORB(T, [x, y], [100., 200.], 1000, 1020) [-1]; \\
& \quad o := [0.3161676404, 0.3184411124] \tag{79} \\
& \text{> } evalf(SS); \\
& \quad [[-0.524740111, 0.1626025726]] \tag{80}
\end{aligned}$$


```

> evalf(o);
[0.3161676404, 0.3184411124] (81)
=
> T := RT([x, y], 30);
T := [ [11 + 17 x + 25 y, 10 + 25 x + 9 y]
[10 + 13 x + 21 y, 7 + 19 x + 23 y] ] (82)
=
> SS := SSg(T, [x, y]);
SS := [[1.211129039, 0.9431816947]] (83)
=
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [1.211129038, 0.9431816947] (84)
=
> evalf(SS);
[[1.211129039, 0.9431816947]] (85)
=
> evalf(o);
[1.211129038, 0.9431816947] (86)
=
> T := RT([x, y], 30);
T := [ [8 + 8 x + 7 y, 20 + 30 x + 16 y]
[9 + 6 x + 18 y, 30 + 15 x + 18 y] ] (87)
=
> SS := SSg(T, [x, y]);
SS := [[0.1322370043, -1.680365877]] (88)
=
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [0.6607006873, 0.9647498239] (89)
=
> evalf(SS);
[[0.1322370043, -1.680365877]] (90)
=
> evalf(o);
[0.6607006873, 0.9647498239] (91)
=
>
=
> T := RT([x, y], 30);
T := [ [26 + 23 x + 3 y, 16 + 11 x + 26 y]
[1 + 11 x + 11 y, 19 + 24 x + 9 y] ] (92)
=
> SS := SSg(T, [x, y]);
SS := [[2.347467067, 0.7461026524]] (93)
=
> o := ORB(T, [x, y], [100., 200.], 1000, 1020)[-1];
o := [2.347467067, 0.7461026535] (94)
=
> evalf(SS);
[[2.347467067, 0.7461026524]] (95)
=
> evalf(o);
[0.6607006873, 0.9647498239] (96)
=
> #Trials 3, 4, 8, 12, 15, 16, and 18.
>

```

Problem 3

read ("DMB12.txt");

For a list of the Main procedures type: Help12(); for help with a specific procedure type: Help12 (ProcedureName); for example Help12(Feig); (1)

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{2 + 4x + 7y + 8z}{1 + 6x + 4y + z}, \frac{7 + 2x + 9y + 5z}{9 + 9x + 4y + 3z}, \frac{4 + 6x + 4y + 4z}{10 + 8x + 2y + 7z} \right]$$

$$SS := [[1.390937686, 0.747936770, 0.659883202]]$$

$$o := [1.390937686, 0.7479367718, 0.6598832011] \quad (2)$$

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{2 + 5x + 3y + 5z}{7 + 5x + 4y + 6z}, \frac{8 + 10x + 2y + 6z}{8 + 4x + 8y + 7z}, \frac{9 + 4x + 10y + 3z}{2 + 9x + y + z} \right]$$

$$SS := [[0.7092008638, 0.898483, 2.398785]]$$

$$o := [0.7092008638, 0.8984818708, 2.398784035] \quad (3)$$

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{5 + 2x + 4y + 4z}{1 + 6x + 3y + z}, \frac{7 + 2x + y + 5z}{10 + 3x + 10y + 4z}, \frac{6 + 8x + 8y + 7z}{9 + 6x + y + 2z} \right]$$

$$SS := [[1.328684296, 0.6749541225, 1.592591199]]$$

$$o := [1.328684301, 0.6749541225, 1.592591199] \quad (4)$$

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{9 + 5x + 3y + 2z}{5 + 3x + 9y + 8z}, \frac{5 + 4x + 3y + 3z}{9 + 3x + 8y + 7z}, \frac{9 + 10x + y + 5z}{9 + x + y + 3z} \right]$$

$$SS := [[0.6956053610, 0.53328988, 1.62970315]]$$

$$o := [0.6956053610, 0.5332898656, 1.629703109] \quad (5)$$

$T := RT([x, y, z], 10);$

$$SS := SSg(T, [x, y, z]);$$

$$o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$$

$$T := \left[\frac{9 + 9x + 9y + 4z}{9 + 2x + 5y + 5z}, \frac{8 + x + 2y + 9z}{5 + 8x + 2y + z}, \frac{9 + 7x + y + 9z}{5 + x + 4y + 8z} \right]$$

$$SS := [[1.555569547, 1.18748623, 1.485421368]]$$

$$o := [1.555569547, 1.187486194, 1.485421355]$$

(6)

$$T := RT([x, y, z], 10);$$

$$SS := SSg(T, [x, y, z]);$$

$$o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$$

$$T := \left[\frac{5 + x + 5y + 10z}{5 + 5x + 5y + 10z}, \frac{3 + 6x + 6y + 7z}{1 + 4x + 2y + 8z}, \frac{2 + 2x + 4y + z}{2 + 5x + 6y + 10z} \right]$$

$$SS := [[0.85603315, 1.867799325, 0.5165009468]]$$

$$o := [0.8560331390, 1.867799326, 0.5165009468]$$

(7)

$$T := RT([x, y, z], 10);$$

$$SS := SSg(T, [x, y, z]);$$

$$o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$$

$$T := \left[\frac{5 + 5x + y + 2z}{8 + 10x + 2y + 5z}, \frac{1 + 3x + 9y + 2z}{2 + 3x + 5y + 4z}, \frac{4 + 5x + 8y + 7z}{1 + 2x + 4y + 2z} \right]$$

$$SS := [[0.4825978712, 0.80370942, 3.0295499]]$$

$$o := [0.4825978712, 0.8037093325, 3.029549846]$$

(8)

$$T := RT([x, y, z], 10);$$

$$SS := SSg(T, [x, y, z]);$$

$$o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$$

$$T := \left[\frac{2 + 3x + 2y + 6z}{4 + 10x + 4y + 4z}, \frac{7 + 5x + y + 5z}{6 + 3x + 8y + 9z}, \frac{9 + 5x + 3y + z}{9 + 8x + 10y + 7z} \right]$$

$$SS := [[0.5791654203, 0.7212300437, 0.5869341863]]$$

$$o := [0.5791654203, 0.7212300430, 0.5869341865]$$

(9)

$$T := RT([x, y, z], 10);$$

$$SS := SSg(T, [x, y, z]);$$

$$o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$$

$$T := \left[\frac{3 + 7x + 3y + 8z}{4 + x + 7y + z}, \frac{6 + 8x + 4y + z}{4 + 10x + 10y + 3z}, \frac{2 + 6x + 10y + z}{6 + 7x + y + 2z} \right]$$

$$\begin{aligned}
SS &:= [[2.3343384, 0.7576616665, 0.980222040]] \\
o &:= [2.334338371, 0.7576616665, 0.9802220435]
\end{aligned} \tag{10}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{9 + 8x + 10y + z}{10 + 7x + 7y + 5z}, \frac{10 + 3x + 5y + 5z}{3 + 4x + 9y + 5z}, \frac{7 + 5x + y + 6z}{10 + 9x + y + 3z} \right] \\
SS &:= [[1.00238249, 1.080848085, 0.79422439]] \\
o &:= [1.002382495, 1.080848085, 0.7942244066]
\end{aligned} \tag{11}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{5 + 10x + 3y + 5z}{9 + 5x + y + 3z}, \frac{2 + 8x + 6y + 2z}{8 + 5x + 9y + z}, \frac{2 + 10x + 10y + 9z}{6 + 6x + 10y + 9z} \right] \\
SS &:= [[0.6139551655, 0.4843774787, -1.546798218]] \\
o &:= [1.327211137, 0.8488721623, 1.041129974]
\end{aligned} \tag{12}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{4 + 9x + 8y + 8z}{7 + 6x + 8y + 9z}, \frac{8 + x + 4y + 8z}{1 + 6x + 3y + 10z}, \frac{9 + 4x + 2y + 8z}{2 + 2x + 7y + 9z} \right] \\
SS &:= [[0.96031417, 1.047659821, 1.120094785]] \\
o &:= [0.9603141253, 1.047659841, 1.120094785]
\end{aligned} \tag{13}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{6 + 7x + 7y + 2z}{2 + 7x + 8y + 3z}, \frac{1 + x + 4y + 10z}{9 + 6x + 9y + 9z}, \frac{3 + 8x + 4y + 3z}{10 + 10x + 9y + 2z} \right] \\
SS &:= [[0.2042499678, -1.329734239, 0.29275959]] \\
o &:= [1.197326906, 0.3880766651, 0.5972384427]
\end{aligned} \tag{14}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{9 + 4x + 2y + 8z}{4 + 3x + 9y + 4z}, \frac{9 + 5x + y + 2z}{9 + 9x + 4y + 2z}, \frac{6 + 7x + 4y + 4z}{4 + 9x + 2y + 9z} \right] \\
SS &:= [[0.1888249963, 0.7340366983, -1.257977342]] \\
o &:= [1.268043397, 0.7115056628, 0.8609702255]
\end{aligned} \tag{15}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{5 + 10x + 7y + 9z}{2 + 7x + y + 4z}, \frac{4 + 6x + y + 8z}{8 + 5x + 6y + 3z}, \frac{3 + 6x + 5y + 7z}{9 + x + 9y + 3z} \right] \\
SS &:= [[1.877839611, 11.52366132, -36.51972956]] \\
o &:= [1.962645110, 0.9684721849, 1.204200682]
\end{aligned} \tag{16}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{8 + 5x + 2y + 5z}{10 + x + 9y + 10z}, \frac{8 + 8x + 2y + 4z}{3 + 10x + 4y + 6z}, \frac{9 + 4x + 2y + 10z}{2 + 3x + 3y + 4z} \right] \\
SS &:= [[0.5803412909, 0.901672319, 2.309326923]] \\
o &:= [0.5803412909, 0.9016723214, 2.309326925]
\end{aligned} \tag{17}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{4 + 6x + 7y + 5z}{2 + 10x + 5y + 2z}, \frac{10 + 3x + 6y + 5z}{9 + x + 4y + 10z}, \frac{3 + 4x + 7y + 7z}{1 + 7x + y + 2z} \right] \\
SS &:= [[1.209277548, -93.940429, 36.944078]] \\
o &:= [1.209957650, 0.8633759381, 1.930074074]
\end{aligned} \tag{18}$$

$$\begin{aligned}
T &:= RT([x, y, z], 10); \\
SS &:= SSg(T, [x, y, z]); \\
o &:= ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];
\end{aligned}$$

$$\begin{aligned}
T &:= \left[\frac{1 + 4x + 7y + 9z}{8 + 9x + 6y + 7z}, \frac{8 + 3x + 3y + 2z}{2 + x + 3y + 5z}, \frac{4 + 3x + 10y + z}{8 + 3x + 8y + 8z} \right] \\
SS &:= [[-2.196782139, 0.9836751024, 0.565862225]] \\
o &:= [0.7341396111, 1.469087218, 0.7704402599]
\end{aligned} \tag{19}$$

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{7 + 5x + 6y + 7z}{1 + 5x + 6y + 6z}, \frac{6 + 4x + 10y + 10z}{5 + 9x + 9y + 5z}, \frac{10 + 9x + 5y + 5z}{2 + 4x + 9y + 10z} \right]$$

$$SS := [[-1.189670387, -0.627019387, 0.9246734834]]$$

$$o := [1.342435875, 1.035927502, 1.166909602]$$

(20)

$T := RT([x, y, z], 10);$
 $SS := SSg(T, [x, y, z]);$
 $o := ORB(T, [x, y, z], [100., 200., 300.], 1000, 1020)[-1];$

$$T := \left[\frac{3 + 7x + y + 4z}{9 + 5x + 9y + 10z}, \frac{7 + 9x + 6y + z}{5 + 2x + y + 3z}, \frac{3 + 10x + 9y + 9z}{2 + 2x + 3y + 10z} \right]$$

$$SS := [[0.3094776489, 1.851627407, 1.550496209]]$$

$$o := [0.3094776486, 1.851627405, 1.550496209]$$

(21)

#Trials 11, 13, 14, 15, 17, 18, and 19.