

hw11

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$$r, \quad a(n+1) = ra(n)(1-a(n))$$

$$0 \leq a(n) \leq 1$$

$$0 \leq r$$

Let $0 \leq x \leq 1$ s.t.

$$x = rx(1-x)$$

$$x = rx - rx^2$$

$$rx^2 + x - rx = 0$$

$$x(rx + 1 - r) = 0$$

$$x = 0$$

$$rx + 1 - r = 0$$

$$x = \frac{r-1}{r}$$

Suppose $0 < r < 1$. Then $\frac{r-1}{r} < 0$ but $0 \leq x$ So $x = 0$ is the only potential steady state.

$$f(x) = rx(1-x) = rx - rx^2$$

$$f'(x) = r - 2rx$$

$$f'(0) = r - 0 = r$$

$$|f'(0)| = |r|$$

So if $|r| < 1$
i.e. $-1 < r < 1$, then 0 is a
stable steady state.Suppose $r = 1$.

$$\frac{r-1}{r} = 0$$

So $x = 0$ is the only
prospective steady state.But $f'(0) = r = 1$ and so 0 is semi-stable.Suppose $r > 1$. Then $\frac{r-1}{r} > 0$ and so valid for $x = \frac{r-1}{r}$.

$$f'\left(\frac{r-1}{r}\right) = r - 2r\left(\frac{r-1}{r}\right) = r - 2r + 2 = 2 - r$$

For $\frac{r-1}{r}$ to be stable, $|f'\left(\frac{r-1}{r}\right)| = |2 - r| < 1$.

$$|2 - r| < 1 \Rightarrow 2 - r < 1, r - 2 < 1$$

$$\Rightarrow 1 < r, r < 3$$

$$\Rightarrow 1 < r < 3$$

So for $1 < r < 3$, $x = \frac{r-1}{r}$ is a stable
steady state.For $r \geq 3$ there are no stable steady states.We observe on Maple that we see a two-cycle $\{a, b, a, b, \dots\}$.So $f(a) = b, f(b) = a, f(f(a)) = a, f(f(b)) = b$.i.e. $f' = f \circ f$ has steady states a and b .

$$f_1(x) = f(f(x)) = f(rx(1-x)) = r(rx(1-x))(1-(rx(1-x)))$$

$$= (r^2x - r^2x^2)(1 - rx + rx^2)$$

$$\begin{array}{ccc} & r^2x & -r^2x^2 \\ 1 & r^2x & -r^2x^2 \\ -rx & -r^3x^2 & r^3x^3 \\ rx^2 & r^3x^3 & -r^3x^4 \end{array}$$

$$\text{So } f_1(x) = -r^3x^4 + 2r^3x^3 - r^3x^2 - r^2x^2 + r^2x.$$

This gives 4 steady states $\rho, \frac{r-1}{r}, \frac{\frac{r}{2} + \frac{1}{2} + \frac{\sqrt{r^2 - 2r - 3}}{2}}{r},$

$$\frac{\frac{r}{2} + \frac{1}{2} - \frac{\sqrt{r^2 - 2r - 3}}{2}}{r}$$

With Maple we can show that for $3 < r < 3.4444897$,
the second two steady states are stable.

If we look at range $3.45 < r < 3.54$, we get a period-4 orbit.

As r increases, the periods of the orbits continue to double.

The "bifurcation" values are the numbers of r at which the period of the orbits f doubles.

Let these be $[a_1, a_2, \dots]$

s.t. for $a_i < r < a_{i+1}$ every orbit converges to an ultimate period of 2^i .

Feigenbaum constant $\lim_{i \rightarrow \infty} \frac{a_{i+1} - a_i}{a_{i+2} - a_{i+1}} = \delta \approx 4.6692.$

$$2. \quad z = \frac{1 + 2z + 3z + 4z}{2 + 3z + 4z + 5z} = \frac{1 + 4z}{2 + 12z}$$

$$2z + 12z^2 = 1 + 4z$$

$$12z^2 - 7z - 1 = 0$$

$$z_1 = \frac{7 + \sqrt{97}}{24} \quad z_2 = \frac{7 - \sqrt{97}}{24}$$

3. Let $\underline{x}(n) = [a(n+2), a(n+1), a(n)]$

$$f(x_1, x_2, x_3) = \frac{1 + 2x_1 + 3x_2 + 4x_3}{2 + 3x_1 + 4x_2 + 5x_3}$$

So $\underline{x}(n+1) = \underline{F}(\underline{x}(n))$

where $\underline{F}$ is the transformation in $\mathbb{R}^3$,

$[x_1, x_2, x_3] \rightarrow \left[\frac{1 + 2x_1 + 3x_2 + 4x_3}{2 + 3x_1 + 4x_2 + 5x_3}, x_1, x_2 \right]$

i.e. $\underline{x}(n) = \begin{pmatrix} a(n+2) \\ a(n+1) \\ a(n) \end{pmatrix}$ ~~$\underline{x}(n+1) = \underline{F}(\underline{x}(n))$~~ $= \begin{pmatrix} f(x_1, x_2, x_3) \\ x_1 \\ x_2 \end{pmatrix}$
 $= \begin{pmatrix} f(a(n+2), a(n+1), a(n)) \\ a(n+2) \\ a(n+1) \end{pmatrix}$

4. SS: $\left[\frac{r}{12} + \frac{5}{12} + \frac{\sqrt{r^2 + 10r + 1}}{12}, \frac{r}{12} + \frac{5}{12} - \frac{\sqrt{r^2 + 10r + 1}}{12} \right]$

$r = 4$

$$f(x) = \frac{(1-2x)(1-3x)}{1-5x+15x^2}$$

$$f'(x) = -\frac{5}{r} + \frac{30x}{r}$$

We need $|f'(s_i)| < 1$ where $s_1 = \frac{r+5+\sqrt{r^2+10r+1}}{12}$
 $s_2 = \frac{r+5-\sqrt{r^2+10r+1}}{12}$

From experimentation, $3 < r < 3.5$

5. i)

~~3.8~~ 3.8

ii)

3.87

iii)

3.9