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yd372.

Homework for Lecture 11 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 13, 2025.

Subject: hw11

with an attachment hw11FirstLast.pdf

1. Read and understand, and be able to reproduce in an exam, the first part of

`http://sites.math.rutgers.edu/~zeilberg/Bio25/L11.pdf`

about period-doubling and advancement to chaos.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) = \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

$\frac{7+\sqrt{97}}{24}$ and $\frac{7-\sqrt{97}}{24}$.

3. Convert the third-order recurrence

$$a(n+3) = \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

$$x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} = \begin{bmatrix} a(n) \\ a(n+1) \\ a(n+2) \end{bmatrix}$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

4. By using `SS` in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1 - 2\,a(n))(1 - 3\,a(n))}{r} \quad .$$

By using `SSS` and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

1. $a(n+1) = r \cdot a(n) \cdot (1 - a(n)) \qquad 0 < r < 1$

$x = r x \cdot (1-x)$

$x = 0 \qquad x = \frac{r-1}{r}$ two steady-states.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} .$$

$z = \frac{1+2z+3z+4z}{2+3z+4z+5z} = \frac{1+9z}{2+12z}$

$\Rightarrow \quad 2z + 12z^2 = 1 + 9z$

solve for: $12z^2 - 7z - 1 = 0$

$z = \frac{7 \pm \sqrt{49+48}}{24} = \frac{7 \pm \sqrt{97}}{24}$

$z_1 = \frac{7+\sqrt{97}}{24}$ and $z_2 = \frac{7-\sqrt{97}}{24}$ are two steady-states.

3. Convert the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} .$$

Let $X_n = \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} = \begin{bmatrix} a_n \\ a(n+1) \\ a(n+2) \end{bmatrix}$

then $x_1(n+1) = x_2(n)$

$x_2(n+1) = x_3(n)$

$x_3(n+1) = \frac{1 + 2x_3(n) + 3x_2(n) + 4x_1(n)}{2 + 3x_3(n) + 4x_2(n) + 5x_1(n)}$

So, $x(n+1) = \begin{bmatrix} x_2(n) \\ x_3(n) \\ \frac{1+2x_3(n)+3x_2(n)+4x_1(n)}{2+3x_3(n)+4x_2(n)+5x_1(n)} \end{bmatrix}$

4. By using SS in the Maple file

http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1-2a(n))(1-3a(n))}{r} \quad .$$

By using SSS and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

Sol. Let $A := a(n+1) = a_n$ non-linear recurrence.

So, $A = \frac{(1-2A)(1-3A)}{r}$

$$Ar = 1-5A+6A^2$$

solve for A, $6A^2 - A(5+r) + 1 = 0$ ... ①

$$A = \frac{5+r \pm \sqrt{(5+r)^2 - 24}}{12} = \frac{5+r \pm \sqrt{r^2+10r+1}}{12}$$

so, two steady-states in term of r is

$$A_1 = \frac{5+r+\sqrt{r^2+10r+1}}{12} \quad \text{and} \quad A_2 = \frac{5+r-\sqrt{r^2+10r+1}}{12}.$$

Let $f(A) = \frac{(1-2A)(1-3A)}{r} = \frac{1-5A+6A^2}{r}$

$$f'(A) = \frac{1}{r}(12A-5) = \frac{12A-5}{r}$$

$|f'(A)| < 1$ since $\Delta = \sqrt{r^2+10r+1} \geq 0 \Rightarrow r < 0$ impossible.

so $f'(A) = 1$ or $f'(A) = -1$

Thus $\frac{12A-5}{r} = -1 \Rightarrow 12A-5 = -r \Rightarrow A = \frac{5-r}{12}$.

substitute into ①, we have.

$$6\left(\frac{5-r}{12}\right)^2 - (5+r)\cdot\frac{5-r}{12} + 1 = 0$$

$$3r^2 - 10r - 1 = 0$$

$\Rightarrow r = \frac{10 \pm \sqrt{112}}{6} = \frac{5 \pm 2\sqrt{7}}{3}$, So $r_0 = \frac{5+2\sqrt{7}}{3} \approx 3.43$ ∎

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

i) $r_1 \approx 3.6$

ii) $r_2 \approx 3.56$

iii) $r_3 \approx 3.569$