Homework for Lecture 11 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 13, 2025.

Subject: hw11

with an attachment hw11FirstLast.pdf

1. Read and understand, and be able to reproduce in an exam, the first part of

`http://sites.math.rutgers.edu/~zeilberg/Bio25/L11.pdf`

about period-doubling and advancement to chaos.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) \,=\, \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

3. Convert the third-order recurrence

$$a(n+3) \,=\, \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

4. By using `SS` in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1\,-\,2\,a(n))(1\,-\,3\,a(n))}{r} \quad .$$

By using `SSS` and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) \;=\; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

$$z = \frac{1+2z+3z+4z}{2+3z+4z+5z}$$

$$z = \frac{1+9z}{2+12z}$$

$$2z+12z^2 = 1+9z$$

$$12z^2-7z-1=0$$

$$z = \frac{7 \pm \sqrt{49-4(12)(-1)}}{2(12)}$$

$$\boxed{z = \frac{7 \pm \sqrt{97}}{24}}$$

potential steady states

$z \approx .702 \,, \; -.119$

3. Convert the third-order recurrence

$$a(n+3) \;=\; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

$$f(x_1, x_2, x_3) = \frac{1+2x_1+3x_2+4x_3}{2+3x_1+4x_2+5x_3}$$

$$x(n+1) = F(x(n))$$

$$[x_1, x_2, x_3] \rightarrow \frac{1+2x_1+3x_2+4x_3}{2+3x_1+4x_2+5x_3}$$

4. By using `SS` in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1-2a(n))(1-3a(n))}{r} \quad .$$

$f := \frac{(1-2\cdot x)\cdot(1-3\cdot x)}{r};$

$$f := \frac{(1-2x)(1-3x)}{r}$$

$SS(f, x);$

$$\left[\frac{r}{12} + \frac{5}{12} + \frac{\sqrt{r^2+10r+1}}{12}, \frac{r}{12} + \frac{5}{12} - \frac{\sqrt{r^2+10r+1}}{12}\right]$$

By using `SSS` and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

$f := \frac{(1-2\cdot x)\cdot(1-3\cdot x)}{3};$

$$f := \frac{(1-2x)(1-3x)}{3}$$

$SS(f, x);$

$$\left[\frac{2}{3} - \frac{\sqrt{10}}{6}, \frac{2}{3} + \frac{\sqrt{10}}{6}\right]$$

$SSS(f, x);$

$$[\,]$$

zero stable steady states

$f := \frac{(1-2\cdot x)\cdot(1-3\cdot x)}{4};$

$$f := \frac{(1-2x)(1-3x)}{4}$$

$SS(f, x);$

$$\left[\frac{3}{4} - \frac{\sqrt{57}}{12}, \frac{3}{4} + \frac{\sqrt{57}}{12}\right]$$

$SSS(f, x);$

$$\left[\frac{3}{4} - \frac{\sqrt{57}}{12}\right]$$

1 stable steady state

$3 < r_0 < 4$

$f := \frac{(1-2\cdot x)\cdot(1-3\cdot x)}{3.4};$

$$f := 0.2941176471\,(1-2x)(1-3x)$$

$SS(f, x);$

$$[0.1313759297, 1.268624070]$$

$SSS(f, x);$

$$[\,]$$

$\boxed{r_0 \approx 3.4}$

$f := \frac{(1-2\cdot x)\cdot(1-3\cdot x)}{3.5};$

$$f := 0.2857142857\,(1-2x)(1-3x)$$

$SS(f, x);$

$$[0.1294815004, 1.287185166]$$

$SSS(f, x);$

$$[0.1294815004]$$

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

$$f := \frac{(1 - 2 \cdot x) \cdot (1 - 3 \cdot x)}{3.3};$$

$$f := 0.3030303030\ (1 - 2\,x)\ (1 - 3\,x)$$

→ period 2

$Orb(f, x, .5, 1000, 1010)$

0.2377713552, 0.04556197821, 0.2377713552, 0.04556197821, 0.2377713552, 0.04556197821, 0.2377713552, 0.04556197821, 0.2377713552, 0.04556197821, 0.2377713552]

$$f := \frac{(1 - 2 \cdot x) \cdot (1 - 3 \cdot x)}{3.4};$$

$$f := 0.2941176471\ (1 - 2\,x)\ (1 - 3\,x)$$

$Orb(f, x, .5, 1000, 1010)$

0.1804738036, 0.08619286610, 0.1804738031, 0.08619286651, 0.1804738026, 0.08619286692, 0.1804738021, 0.08619286737, 0.1804738016, 0.08619286778, 0.1804738011] aperiodic?