

Homework for Lecture 11 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 13, 2025.

Subject: hw11

with an attachment hw11FirstLast.pdf

1. Read and understand, and be able to reproduce in an exam, the first part of

<http://sites.math.rutgers.edu/~zeilberg/Bio25/L11.pdf>

about period-doubling and advancement to chaos.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} .$$

3. Convert the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} .$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

4. By using SS in the Maple file

<http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf>

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1 - 2a(n))(1 - 3a(n))}{r} .$$

By using SSS and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one there is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0 .

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing ‘high-low’ estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing ‘high-low’ estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)}$$

$$z = \frac{1+2z+3z+4z}{2+3z+4z+5z}$$

$$z = \frac{1+9z}{z+12z} \quad \begin{cases} z(2+12z) - (1+9z) = 0 \\ 12z^2 - 7z - 1 = 0 \end{cases}$$

$$z = \frac{7 \pm \sqrt{49 - 4(12)(-1)}}{24}$$

$$z = \frac{7 \pm \sqrt{97}}{24}$$

3. Convert the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)}$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

$$\mathbf{x}(n) = [a(n+2), a(n+1), a(n)]$$

$$f(x_1, x_2, x_3) = \frac{1 + 2x_1 + 3x_2 + 4x_3}{2 + 3x_1 + 4x_2 + 5x_3}$$

$$\mathbf{x}(n+1) = f(\mathbf{x}(n))$$

$$[x_1, x_2, x_3] \rightarrow \left[\frac{1 + 2x_1 + 3x_2 + 4x_3}{2 + 3x_1 + 4x_2 + 5x_3}, x_1, x_2 \right]$$

4. By using SS in the Maple file

<http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf>

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1 - 2a(n))(1 - 3a(n))}{r}$$

By using SSS and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0 .

```
SS(f,x)
[ r/12 + 5/12 + sqrt(r^2+10r+1)/12, r/12 + 5/12 - sqrt(r^2+10r+1)/12 ]

> f1 := ((1-2*x)/(1-3*x))/3
f1 := (1-2*x)/(1-3*x)/3

> SSS(f1,x)
[ ]

> f1 := ((1-2*x)/(1-3*x))/4
f1 := (1-2*x)/(1-3*x)/4

> SSS(f1,x)
[ 3/4 - sqrt(37)/12 ]
```

$$r_0 = 4$$

4

```

> f1 := ((1 - 2 * x) * (1 - 3 * x)) / 3.430501
f1 := 0.2915026114261444611151549001151726817744696765865977010355047265690929692193647516791279174674486321 (1 - 2 x) (1 - 3 x)
> SSS(f1, x)
[0.1307915914286220477031677803555480104046585372699992185518465754182228167383893464447532459764528769]
> f1 := ((1 - 2 * x) * (1 - 3 * x)) / 3.43050
f1 := 0.2915026963999416994607200116601078559976679784288004664043142399067191371520186561725695962687654861 (1 - 2 x) (1 - 3 x)
> SSS(f1, x)
[]

```

 $r_0 = 3.4305$

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

```

> f1 := ((1 - 2 * x) * (1 - 3 * x)) / 1.515065
f1 := 0.6600376881519934788276410583044291829063439522396728853217518720318930210915043248969516159372700181 (1 - 2 x) (1 - 3 x)
> SSS(f1, x)
[]
> evalf(Orb(f1, x, 0.9, 1000, 1010), 10);
[0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905]
> f1 := ((1 - 2 * x) * (1 - 3 * x)) / 1.51507
f1 := 0.6600355099104331813051542172968905727128119492828714184823143485119499429069283927475298171041602038 (1 - 2 x) (1 - 3 x)
> SSS(f1, x)
[]
> evalf(Orb(f1, x, 0.9, 1000, 1010), 10);
[0.7534432132, 0.4216597166, -0.02740274932, 0.7534432145, 0.4216597204, -0.02740274913, 0.7534432132, 0.4216597166, -0.02740274932, 0.7534432145]

```

I could only find 4 → 8