Homework for Lecture 11 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (as a .pdf file) to

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by 8:00pm Monday, Oct. 13, 2025.

Subject: hw11

with an attachment hw11FirstLast.pdf

1. Read and understand, and be able to reproduce in an exam, the first part of

`http://sites.math.rutgers.edu/~zeilberg/Bio25/L11.pdf`

about period-doubling and advancement to chaos.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) \;=\; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

3. Convert the third-order recurrence

$1-5x+6x^2$

$$a(n+3) \;=\; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

4. By using `SS` in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1\,-\,2\,a(n))(1\,-\,3\,a(n))}{r} \quad .$$

By using `SSS` and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

2. Find the potential steady-states of the third-order recurrence

$$a(n+3) \; = \; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

$$z = \frac{1+2z+3z+4z}{2+3z+4z+5z}$$

$$z = \frac{1+9z}{2+12z}$$

$$z(2+12z)-(1+9z)=0$$

$$12z^2-7z-1=0$$

$$z = \frac{7 \pm \sqrt{49-4(12)(-1)}}{24}$$

$$z = \frac{7 \pm \sqrt{97}}{24}$$

3. Convert the third-order recurrence

$$a(n+3) \; = \; \frac{1+2a(n+2)+3a(n+1)+4a(n)}{2+3a(n+2)+4a(n+1)+5a(n)} \quad .$$

to a first-order vector recurrence for an appropriate $\mathbf{x}(n)$ (you first have to define it).

$$x(n) = [a(n+2), a(n+1), a(n)]$$

$$f(x_1, x_2, x_3) = \frac{1+2x_1+3x_2+4x_3}{2+3x_1+4x_2+5x_3}$$

$$x(n+1) = F(x(n))$$

$$[x_1, x_2, x_3] \rightarrow \left[\frac{1+2x_1+3x_2+4x_3}{2+3x_1+4x_2+5x_3}, x_1, x_2\right]$$

4. By using `SS` in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1-2\,a(n))(1-3\,a(n))}{r} \quad .$$

By using `SSS` and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

```
SS(f, x)
```

$$\left[\frac{r}{12}+\frac{5}{12}+\frac{\sqrt{r^2+10r+1}}{12}, \frac{r}{12}+\frac{5}{12}-\frac{\sqrt{r^2+10r+1}}{12}\right]$$

```
> f1 := ((1 - 2·x)·(1 - 3·x))/3
```

$$f1 := \frac{(1-2x)(1-3x)}{3}$$

```
> SSS(f1, x)
```

$$[\,]$$

```
> f1 := ((1 - 2·x)·(1 - 3·x))/4
```

$$f1 := \frac{(1-2x)(1-3x)}{4}$$

```
> SSS(f1, x)
```

$$\left[\frac{3}{4}-\frac{\sqrt{57}}{12}\right]$$

$$r_0 = 4$$

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

```
> f1 := ((1 - 2·x)·(1 - 3·x))/3.430501
```

$$f1 := 0.29150261142614446111515490011517268177446967658659770103550472656909296921936475167912791746744486321\,(1-2x)\,(1-3x)$$

```
> SSS(f1, x)
```

$$[0.13079159142862204770316778035554801040465853726999921855184657541822281673838934644475324597645287\,69]$$

```
> f1 := ((1 - 2·x)·(1 - 3·x))/3.43050
```

$$f1 := 0.29150269639994169946072001166010785599766797842880046640431423990671913715201865617256959626876548\,61\,(1-2x)\,(1-3x)$$

```
> SSS(f1, x)
```

$$[\,]$$

$$r_0 = 3.4305$$

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

```
> f1 := ((1 - 2·x)·(1 - 3·x))/1.515065
        f1 := 0.6600376881519934788276410583044291829063439522396728853217518720318930210915043248969516159372700181 (1 - 2x) (1 - 3x)
> SSS(f1, x)
        [ ]
> evalf(Orb(f1, x, 0.9, 1000, 1010), 10);
[0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905, -0.02740258359, 0.7534451183, 0.4216661905, -0.02740258359]
```

```
> f1 := ((1 - 2·x)·(1 - 3·x))/1.51507
        f1 := 0.6600355099104331813051542172968905727128119492828714184823143485119499429069283927475298171041602038 (1 - 2x) (1 - 3x)
> SSS(f1, x)
        [ ]
> evalf(Orb(f1, x, 0.9, 1000, 1010), 10);
[0.7534432132, 0.4216597166, -0.02740274932, 0.7534432145, 0.4216597204, -0.02740274913, 0.7534432132, 0.4216597166, -0.02740274932, 0.7534432145, 0.4216597204]
```

the only one I could find was $4 \to 8$